MATHEMATICS

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Abstract

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ASYMPTOTIC REPRESENTATIONS AND SOME ESTIMATES FOR INTEGRAL ERROR FUNCTIONS OF ARBITRARY ORDER

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The solution of many problems of mathematical physics is conveniently expressed through $i^{\mu}\operatorname{erfc} z$—integral error functions of order μ, which are solutions of the equation

$$w'' + 2zw' - 2\mu w = 0 \tag{1}$$

and, for $\operatorname{Re}\mu > -1$, admit the representation in the form of the integral

$$i^{\mu}\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\frac{e^{-z^2}}{\Gamma(1+\mu)}\int_0^{\infty} t^{\mu} e^{-2zt-t^2}\,dt. \tag{2}$$

Here we shall consider some properties of the function $i^{\mu}\operatorname{erfc} z$, or, for brevity, $i_{\mu}(z)$, for real values of the index and argument.

1. $\mu = \nu > -1$; $z = x \geq 0$. This case has been treated in detail in ($^{1-3}$), where expressions have been obtained both for small and for large x:

$$i_{\nu}(x) = \frac{e^{-x^2}}{\sqrt{\pi}\,\Gamma(1+\nu)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(2x)^k\Gamma\left(\frac{1+\nu+k}{2}\right); \tag{3}$$

$$i_{\nu}(x) = \frac{2e^{-x^2}}{\sqrt{\pi}\,(2x)^{\nu+1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(1+\nu+2k)}{\Gamma(1+\nu)}(2x)^{-2k}. \tag{4}$$

If both the argument and the index increase simultaneously, while $\nu/x\sqrt{2} \ll 1$, then, using the method set forth in (4), we obtain

$$i_{\nu}(x) \sim \frac{2e^{-x^2-\nu^2/4x^2}}{\sqrt{\pi}\,(2x)^{\nu+1}}\left[1 + \frac{1}{6\nu}\left(1 - 11\frac{\nu^2}{2x^2} + 3\frac{\nu^4}{4x^4}\right)\right]. \tag{5}$$

From the recurrence relations

$$2\nu i_\nu(x) = -2x i_{\nu-1}(x) + i_{\nu-2}(x); \tag{6}$$

$$i'_\nu(x) = -i_{\nu-1}(x) \tag{7}$$

we estimate the modulus of the logarithmic derivative $i_{\nu-1}(x)/i_\nu(x)$. From (6) we have

$$\frac{i_{\nu-1}(x)}{i_\nu(x)} = 2x + \frac{2(\nu+1)}{2x + i_\nu(x)/i_{\nu+1}(x)} \le 2x + \frac{2(\nu+1)}{2x}.$$

On the other hand,

$$\frac{i_\nu(x)}{i_{\nu+1}(x)} = 2x + \cfrac{2(\nu+2)}{2x + \cfrac{2(\nu+3)}{2x + i_{\nu+2}(x)/i_{\nu+3}(x)}} \ge 2x + \cfrac{2(\nu+2)}{2x + \cfrac{2(\nu+3)}{2x}},$$

whence, for $(2x)^2 \ge (\nu+1)(\nu+3)$, we obtain

$$i_\nu^2(x) \ge i_{\nu-1}(x) i_{\nu+1}(x) \quad \text{for } \nu \ge -1. \tag{8}$$

2. $\mu = \nu > -1;\ z = -x \leqslant 0$. In this case

$$i_\nu(-x) = \frac{2}{\sqrt{\pi}\Gamma(1+\nu)} e^{-x^2} \int_0^\infty t^\nu e^{2xt-t^2}\, dt = \frac{2}{\sqrt{\pi}\Gamma(1+\nu)} x^\nu \int_{-x}^\infty \left[1 + \frac{\tau}{x}\right]^\nu e^{-\tau^2}\, d\tau. \tag{9}$$

Writing $e^{2xt} = 1 + 2xt + (2xt)^2/2! + \cdots$, we obtain

$$i_\nu(-x) = \frac{1}{\sqrt{\pi}\Gamma(1+\nu)} e^{-x^2} \sum_{k=0}^\infty \frac{(2x)^k}{k!} \Gamma\left(\frac{1+\nu+k}{2}\right). \tag{10}$$

The asymptotic expansion of $i_\nu(-x)$ for large x will be found from (9), if we use the facts that

$$\int_{-x}^{+x} \tau^{2s+1} e^{-\tau^2}\, d\tau = 0 \quad \text{and} \quad \int_{-x}^{+x} \tau^{2s} e^{-\tau^2}\, d\tau = \frac{\sqrt{\pi}(2s)!}{s!\, 4^s} + O(e^{-x^2}).$$

Then

$$i_{+\nu}(-x) = \frac{2}{\Gamma(1+\nu)} x^{\nu} \left\{ \sum_{k=0}^{N-1} \frac{\Gamma(1+\nu)}{k!\,\Gamma(1+\nu-2k)} (2x)^{-2k} + O[(2x)^{-2N}] + R_{\nu} \right\}, \tag{11}$$

where $R_{\nu} < \sqrt{\pi/2}\,(2/x)^{\nu}\operatorname{erfc}(x/\sqrt{2})$.

For integral ν the series in (11) terminates and becomes, for $\nu = 2k$,

$$i_{2k}(-x) = \frac{2}{(2k)!} x^{2k} \left\{ \sum_{s=0}^{k} \frac{(2k)!}{s!(2k-2s)!} (2x)^{-2s} + R_{2k} \right\} \tag{11a}$$

and for $\nu = 2k+1$,

$$i_{2k+1}(-x) = \frac{2}{(2k+1)!} x^{2k+1} \left\{ \sum_{s=0}^{k} \frac{(2k+1)!}{s!(2k+1-2s)!} (2x)^{-2s} + R_{2k+1} \right\}. \tag{11b}$$

Further, from (6) we have

$$\frac{i_{\nu-1}(-x)}{i_{\nu}(-x)} = \frac{2\nu}{2x + i_{\nu-2}(-x)/i_{\nu-1}(-x)} \leqslant \frac{2\nu}{2x},$$

since for $\nu \geqslant 1$, $i_{\nu-2}(-x)/i_{\nu-1}(-x) > 0$. On the other hand,

$$\frac{i_{\nu}(-x)}{i_{\nu+1}(-x)} = \frac{2(\nu+1)}{2x + \cfrac{2\nu}{2x + i_{\nu-2}(-x)/i_{\nu-1}(-x)}} \geqslant \frac{2(\nu+1)}{2x + \cfrac{2\nu}{2x}},$$

whence, for $(2x)^2 \geqslant 2\nu^2$, we obtain

$$i_{\nu}^2(-x) \geqslant i_{\nu-1}(-x) i_{\nu+1}(-x). \tag{12}$$

We note that (12) is also valid for $\nu \geqslant -1$, which is established directly from (9) and (11).

We turn to functions with negative index less than -1. Using the analytic continuation of $\Gamma(1+\mu)$ into the region $\operatorname{Re}\mu \leqslant -1$*, we shall everywhere in the coefficients of the expansions of paragraphs 1 and 2 replace $+\nu$ by $-\nu$. Then we obtain results for the following case:

3. $\mu = -\nu \leqslant -1$; $z = x \geqslant 0$. From expressions (3) and (4) we obtain

$$i_{-\nu}(x)=\frac{1}{\sqrt{\pi}\Gamma(1-\nu)}e^{-x^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(2x)^k\Gamma\left(\frac{1-\nu+k}{2}\right); \tag{13}$$

$$i_{-\nu}(x)=\frac{2e^{-x^2}}{\sqrt{\pi}}(2x)^{\nu-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(2x)^{-2k}\frac{\Gamma(1-\nu+2k)}{\Gamma(1-\nu)}. \tag{14}$$

* For all complex μ (5)

$$\Gamma(1+\mu)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{1+\mu+n}+\int_1^{\infty}t^{\mu}e^{-t}\,dt.$$

For integer ν, the series in (13) and (14) terminate because of the poles of the Γ-functions, and we obtain the well-known relation between $i_{-n}(x)$ and the Hermite polynomial

$$i_{-n}(x)=\frac{2}{\sqrt{\pi}}e^{-x^2}H_{n-1}(x). \tag{15}$$

Using the asymptotic expansion (14) for $2x>\nu+1$, one may obtain

$$i_{-\nu}^2(x)\geq i_{-\nu-1}(x)i_{-\nu+1}(x)\quad\text{for }-\nu\leq-1. \tag{16}$$

4. $\mu=-\nu\leq-1;\ z=-x\leq 0$. In this case the expressions (10) and (11) of item 2, after changing the sign of ν, become

$$i_{-\nu}(-x)=\frac{1}{\sqrt{\pi}\Gamma(1-\nu)}e^{-x^2}\sum_{k=0}^{\infty}\frac{(2x)^k}{k!}\Gamma\left(\frac{1-\nu+k}{2}\right); \tag{17}$$

$$i_{-\nu}(-x)=\frac{2x^{-\nu}}{\Gamma(1-\nu)}\left\{\sum_{k=0}^{N-1}\frac{\Gamma(1-\nu)}{k!\Gamma(1-\nu-2k)}(2x)^{-2k}+O\left[(2x)^{-2N}\right]+R_{-\nu}\right\}, \tag{18}$$

where $R_{-\nu}>\sqrt{\pi}\,2^{-(1+\nu)}\operatorname{erfc}x$.

For integer ν, (17) also reduces to (15), but now for a negative argument. The asymptotic behavior of $i_{-n}(-x)$ for integer n can be found from (14), replacing there x by $-x$: for large x, $i_{-n}(-x)$ tends to zero as

$$i_{-n}(-x)\sim\frac{2}{\sqrt{\pi}}e^{-x^2}(-2x)^{n-1}. \tag{19}$$

As for fractional ν, it follows from (18) that $i_{-\nu}(-x)$ asymptotically tends to zero as

$$i_{-\nu}(x) \sim \frac{2}{\Gamma(1-\nu)} x^{-\nu}. \tag{20}$$

We note that, because of the pole of $\Gamma(1-n)$ for integer n, for $i_{-n}(-x)$ we immediately obtain 0 from (18) and (20). From the same relations we find that, asymptotically, $i_{-\nu-1}(-x)/i_{-\nu}(-x) \sim -\nu/x$ for $(2x)^2 > (\nu+1)(\nu+2)$, whence

$$i^2_{-\nu}(-x) \geq i_{-\nu-1}(-x) i_{-\nu+1}(-x), \quad \nu \text{ noninteger } \geq 1. \tag{21}$$

If $\nu = n$ (n integer), then from (16) and (18), for $2x > n+1$, we find $i_{-n-1}(-x)/i_{-n}(-x) = -i_{-n-1}(x)/i_{-n}(x) \sim -2x$, whence

$$i^2_{-n}(-x) \leq i_{-n-1}(-x) i_{-n+1}(-x). \tag{22}$$

Further, under the corresponding restrictions on x from (6), (8), and (12), we obtain

$$\frac{\nu}{\nu+1} \leq \frac{i_{\nu-1}(x) i_{\nu+1}(x)}{i^2_{\nu}(x)} \leq 1 \quad \text{for } \nu \geq -1. \tag{23}$$

In conclusion we note that relations analogous to (16), (21), (22) can also be obtained for the functions $H_n(x)$ and $D_\nu(x)$, if one uses (15) and the expression

$$i_\nu(x) = \frac{2}{\sqrt{\pi}} e^{-x^2/2} \cdot 2^{-\frac{\nu+1}{2}} D_{-\nu-1}(x\sqrt{2}).$$

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CITED LITERATURE

1. E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, 2, 1934.

2. W. Magnus, F. Oberhettinger, *Formeln und Sätze für die speziellen Funktionen*, Leipzig, 1948.

3. O. S. Berlyand, L. A. Sokolovskaya, *Engineering-Physics Journal*, 1, No. 11 (1958).

4. A. Ya. Pressman, O. S. Berlyand, *DAN*, 126, No. 3, 508 (1959).

5. V. I. Smirnov, *A Course of Higher Mathematics*, 3, 1950.

Note: Figure translations are in progress. See original paper for figures.