



---

Soviet-era science, translated into English

# Reports of the Academy of Sciences of the USSR

1961

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196101.34034>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**Reports of the Academy of Sciences of the USSR**

1961. Vol. 141, No. 3

**MATHEMATICS**

**M. K. POTAPOV**

**ON THE FOURIER COEFFICIENTS OF PERIODIC FUNCTIONS BELONGING TO THE  $H$ -CLASSES OF S. M. NIKOLSKII**

*(Presented by Academician A. N. Kolmogorov on 23 VI 1961)*

I. As usual <sup>(1)</sup>, we shall say that a function  $f(x)$ , periodic with period  $2\pi$ , belongs to the class  $H_p^{(r)}$ , where  $1 \leq p \leq \infty$ ,  $r = \bar{r} + \alpha$ ,  $0 < \alpha \leq 1$ ,  $\bar{r}$  is an integer, if  $f(x)$  has an  $\bar{r}$ -th derivative such that  $f^{(\bar{r})}(x) \in L_p$  and

$$\|f^{(\bar{r})}(x+h) - 2f^{(\bar{r})}(x) + f^{(\bar{r})}(x-h)\|_{L_p} \leq |h|^\alpha,$$

where in this inequality, for  $0 < \alpha < 1$ , the second difference may be replaced by the first. By  $E_n(f)_{L_p}$  we shall denote the best approximation, in the metric  $L_p$ , of the periodic function  $f(x)$  by trigonometric polynomials of order  $n$ .

It is well known <sup>(1)</sup>:

**Theorem A.** *In order that  $f(x) \in H_p^{(r)}$ , it is necessary and sufficient that  $E_n(f)_{L_p} \leq C/n^r$ .*

Since we are dealing with periodic functions, it would naturally be desirable to pose the question of the existence of conditions, simultaneously necessary and sufficient for  $f(x) \in H_p^{(r)}$ , not in terms of best approximations, but in terms of the Fourier coefficients of the function  $f(x)$ . For even and odd functions such conditions are known <sup>(2,3)</sup>, namely: if

$$f(x) \sim \sum_{k=1}^{\infty} a_k \cos kx$$

or

$$f(x) \sim \sum_{k=1}^{\infty} a_k \sin kx$$

and  $a_k \downarrow$ , then the conditions

$$|a_k| \leq C/k^{r+1-1/p}$$

are necessary and sufficient in order that  $f(x) \in H_p^{(r)}$ ,  $1 < p \leq \infty$ . From Theorems 3, 4, and 5 given below it follows, in particular, that in the general case no such conditions exist.

**II. Theorem 1.** *In order that a periodic function*

$$f(x) \sim \sum_{k=-\infty}^{\infty} C_k e^{ikx}$$

*belong to the class  $H_p^{(r)}$ ,  $1 \leq p \leq \infty$ , it is sufficient that its Fourier coefficients satisfy the following conditions:*

$$\text{for } 2 \leq p \leq \infty \quad \left( \sum_{|k|=n+1}^{\infty} |C_k|^{p'} \right)^{1/p'} \leq \frac{C}{n^r}, \quad \frac{1}{p} + \frac{1}{p'} = 1; \quad (1)$$

$$\text{for } 1 \leq p \leq 2 \quad \left( \sum_{|k|=n+1}^{\infty} |C_k|^2 \right)^{1/2} \leq \frac{C}{n^r}. \quad (2)$$

**Proof.** Obviously, for any  $p$ ,  $1 \leq p \leq \infty$ ,

$$E_n(f) \leq \|f(x) - s_n(x)\|_{L_p} *.$$

Let  $2 \leq p \leq \infty$ ; then, on the basis of the theo—

---

\* Here and below,  $s_n(x)$  denotes the partial sum of the Fourier series for  $f(x)$ . by the Hausdorff–Young theorem ((4), p. 131), it follows from this inequality that

$$E_n(f)_{L_p} \leq \left( \sum_{|k|=n+1}^{\infty} |C_k|^{p'} \right)^{1/p'}. \quad (3)$$

Let  $1 \leq p \leq 2$ ; then, obviously,

$$E_n(f)_{L_p} \leq CE_n(f)_{L_2} = C \left( \sum_{|k|=n+1}^{\infty} |C_k|^2 \right)^{1/2}; \quad (4)$$

if now  $f(x)$  satisfies the conditions of Theorem 1, then from inequalities (3) or (4) and Theorem A there follows the validity of Theorem 1. We note that Theorem 1 cannot be strengthened (see below the example to Theorem 3).

**Theorem 2.** If a periodic function

$$f(x) \sim \sum_{k=-\infty}^{\infty} C_k e^{ikx}$$

belongs to the class  $H_p^{(r)}$ ,  $1 < p \leq \infty$ , then its Fourier coefficients must satisfy the following conditions:

$$\text{for } 2 \leq p \leq \infty \quad \left( \sum_{|k|=n+1}^{\infty} |C_k|^2 \right)^{1/2} \leq \frac{C}{n^r}; \quad (5)$$

$$\text{for } 1 < p \leq 2 \quad \left( \sum_{|k|=n+1}^{\infty} |C_k|^{p'} \right)^{1/p'} \leq \frac{C}{n^r}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (6)$$

**Proof.** It is easy to show (see, for example, (5), p. 77) that for  $1 < p < \infty$

$$\|f(x) - s_n(x)\|_{L_p} \leq CE_n(f)_{L_p}.$$

Therefore, if  $f(x) \in H_p^{(r)}$ , then for  $1 < p \leq 2$

$$\|f(x) - s_n(x)\|_{L_p} \leq C/n^r,$$

whence, on the basis of the Hausdorff–Young theorem, there follows the validity of inequality (6). Let now  $2 \leq p \leq \infty$ . Then, obviously,

$$E_n(f)_{L_2} \leq CE_n(f)_{L_p},$$

whence, on the basis of Parseval's equality and the conditions of the theorem, there follows the validity of inequality (5). The theorem is proved. We note that Theorem 2 cannot be sharpened (see below the examples to Theorems 4 and 5).

As a consequence of Theorem 1, the following is proved quite simply.

**Theorem 3.** In order that a periodic function

$$f(x) \sim \sum_{k=-\infty}^{\infty} C_k e^{ikx}$$

belong to the class  $H_p^{(r)}$ ,  $1 \leq p \leq \infty$ , it is sufficient that its Fourier coefficients satisfy the following conditions:

$$\text{for } 1 \leq p \leq 2 \quad |C_k| \leq C/(|k| + 1)^{r+1/2}; \quad (7)$$

$$\text{for } 2 \leq p \leq \infty \quad |C_k| \leq C/(|k| + 1)^{r+1-1/p}. \quad (8)$$

Let us give examples showing that Theorem 3 cannot be strengthened.

**Case**  $2 \leq p \leq \infty$ . Consider

$$f(x) \sim \sum_{n=1}^{\infty} \frac{\cos(nx - r\pi/2)}{n^{r+1-1/p}}.$$

The function  $f(x)$  satisfies the conditions of Theorem 3; therefore it belongs to the class  $H_p^{(r)}$ . We shall show that  $f(x)$  cannot have a derivative in the sense of Weyl of order  $r$  belonging to  $L_p$ , i.e.  $f(x) \notin W_p^{(r)}$ . Hence it will follow that  $f(x)$  cannot belong to the better class  $H_p^{(\rho)}$ , where  $\rho > r$ , since  $H_p^{(\rho)} \subset W_p^{(r)} \subset H_p^{(r)}$ .

Suppose the contrary—that  $f(x)$  has a derivative of order  $r$  belonging to  $L_p$ , i.e.  $f^{(r)}(x) \in L_p$ ; then we would have

$$\begin{aligned} f^{(r)}(x) &\sim \\ &\sim \sum_{k=1}^{\infty} a_k \cos kx, \quad \text{where } a_k = \frac{1}{k^{1-1/p}}. \end{aligned}$$

By assumption,  $f^{(r)}(x)$  must belong to  $L_p$ . But in order that, for  $1 < p < \infty$ , the function

$$\varphi(x) \sim \sum_{k=1}^{\infty} a_k \cos kx, \quad \text{where } a_1 \geq a_2 \geq \dots \rightarrow 0,$$

belong to  $L_p$ , it is necessary and sufficient ((4), p. 212) that the expression

$$\sum_{k=1}^{\infty} a_k^p k^{p-2}$$

be finite; in our case it is infinite. Hence, for  $2 \leq p < \infty$  our assumption is false and the theorem cannot be strengthened.

For  $p = \infty$ , the function  $f^{(r)}(x) = -\log 2 \sin \frac{1}{2}x$ , and this function does not belong to  $L_\infty$ ; hence, for  $p = \infty$  too, our assumption is false and the theorem cannot be strengthened.

**Case  $1 \leq p \leq 2$ .** Consider

$$f(x) \sim \sum_{k=1}^{\infty} \frac{\varepsilon_k}{k^{r+1/2}} \cos \left( nx - r \frac{\pi}{2} \right),$$

where  $\varepsilon_k = \pm 1$  and the plus and minus signs are chosen so that the series

$$\sum_{k=1}^{\infty} \pm \frac{\cos kx}{k^{1/2}}$$

is not a Fourier series. The function  $f(x)$  satisfies the conditions of Theorem 3, and therefore it belongs to the class  $H_p^{(r)}$ . To show that Theorem 3 cannot be strengthened, it suffices, as above, to show that the function  $f(x)$  cannot have a derivative of order  $r$  belonging to  $L_p$ . Assuming the contrary, we would have  $f^{(r)}(x) \in L_p$  and

$$f^{(r)}(x) \sim \sum_{k=1}^{\infty} \pm \frac{\cos kx}{k^{1/2}}. \quad (9)$$

Since the series (9) is not a Fourier series, our assumption about the existence of  $f^{(r)}(x) \in L_p$  is false, and this means that the theorem cannot be strengthened.

**Theorem 4.** If the periodic function

$$f(x) \sim \sum_{k=-\infty}^{\infty} C_k e^{ikx}$$

belongs to the class  $H_p^{(r)}$ , then its Fourier coefficients necessarily satisfy the conditions

$$|C_k| \leq C/(|k| + 1)^r, \quad |k| = 0, 1, 2, \dots \quad (10)$$

Theorem 4 is known (see, for example, (4), p. 22); it is included here for completeness of exposition. Theorem 4 cannot be sharpened. Indeed, the function

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^{kr}} \cos \left( 2^{kx} + r \frac{\pi}{2} \right)$$

satisfies the conditions of Theorem 4, i.e.  $|C_k| \leq C/(|k| + 1)^r$  and  $f(x) \in H_p^{(r)}$ , but  $f(x) \notin W_p^{(r)}$ .

**Remark.** Assuming that  $|C_k| \downarrow$ , Theorem 4 can be sharpened (see Theorem 5). However, as the examples to Theorem 3 show, the assumption that  $|C_k| \downarrow$  does not allow Theorem 3 to be sharpened.

**Theorem 5.** If the periodic function

$$f(x) \sim \sum_{k=-\infty}^{\infty} C_k e^{ikx}$$

belongs to the class  $H_p^{(r)}$ ,  $1 \leq p \leq \infty$ , and

$$2|C_0| \geq |C_1| + |C_{-1}| \geq |C_2| + |C_{-2}| \geq \dots,$$

i.e.  $|C_k| \downarrow$ , then its Fourier coefficients necessarily satisfy the following conditions:

$$\text{for } 2 \leq p < \infty \quad |C_k| \leq C/(|k| + 1)^{r+1/2}; \quad (11)$$

$$\text{for } 1 \leq p \leq 2 \quad |C_k| \leq C/(|k| + 1)^{r+1-1/p}. \quad (12)$$

**Proof.** Let  $|C_k| \downarrow$ , i.e.

$$2|C_0| \geq |C_1| + |C_{-1}| \geq |C_2| + |C_{-2}| \geq \dots;$$

then

$$\sum_{k=[n/2]+1}^n \frac{|C_k|}{|k|} \geq (|C_n| + |C_{-n}|) \left( \frac{1}{[n/2] + 1} + \dots + \frac{1}{n} \right) \geq \frac{|C_n| + |C_{-n}|}{2},$$

whence

$$\begin{aligned} & |C_n| + |C_{-n}| \leq \\ & \leq 2 \sum_{k=[n/2]+1}^{\infty} \frac{|C_k|}{|k|} \leq 2 \left( \sum_{|k|=[n/2]+1}^{\infty} \frac{1}{|k|^p} \right)^{1/p} \left( \sum_{|k|=[n/2]+1}^{\infty} |C_k|^{p'} \right)^{1/p'}, \quad (13) \\ & \frac{1}{p} + \frac{1}{p'} = 1. \end{aligned}$$

Let the function  $f(x)$  satisfy the conditions of Theorem 5. Then for  $1 < p \leq 2$  the validity of Theorem 5 follows from inequality (13) and Theorem 2, while for  $2 \leq p \leq \infty$  the validity of Theorem 5 follows from inequality (13), if in it one puts  $p = p' = 2$ , and also from Theorem 2. For  $p = 1$ , Theorem 5 is a particular case of Theorem 4.

We give examples showing that Theorem 5 cannot be sharpened.

**Case  $2 \leq p \leq \infty$ .** Consider

$$f(x) = \sum_{n=2}^{\infty} \frac{e^{in \ln n}}{n^{r+1/2}} e^{inx}.$$

This function has Fourier coefficients satisfying inequality (11) and such that  $|C_k| \downarrow$ . We shall show that  $f(x) \in H_p^{(r)}$  for every  $p$ .

Indeed, for  $p = \infty$  the function  $f(x) \in H_\infty^{(r)}$ , for if  $r = \bar{r} + \alpha$ , where  $0 < \alpha \leq 1$ , and  $\bar{r}$  is an integer, then

$$f^{(\bar{r})}(x) = \sum_{n=2}^{\infty} \frac{e^{in \ln n}}{n^{\alpha+1/2}} e^{inx} \in H_\infty^{(\alpha)}$$

(see (4), p. 140)\*, i.e.  $f(x) \in H_p^{(r)}$  for every  $p$ . We further show that  $f(x) \notin W_p^{(r)}$ . Assuming the contrary, we obtain that  $f^{(r)}(x) \in L_p$  and

$$f^{(r)}(x) \sim \sum_{n=2}^{\infty} \frac{e^{in \ln n}}{n^{1/2}} e^{inx}.$$

However, this is not so, since for  $p = \infty$ ,  $f^{(r)}(x) \notin L_\infty$  (see (4), p. 122), and for  $p = 2$ ,  $\|f^{(r)}(x)\|_{L_2} = \infty$ , i.e.  $f^{(r)}(x) \notin L_2$ . This means that  $f^{(r)}(x) \notin L_p$  for all  $p$  in the interval  $2 \leq p \leq \infty$ , i.e. Theorem 5 in this case is not sharpenable.

**Case  $1 < p \leq 2$ .** Consider

$$f(x) \sim \sum_{k=1}^{\infty} \frac{\cos(kx - r\pi/2)}{k^{r+1-1/p}}.$$

This function has Fourier coefficients satisfying inequality (12) and such that  $|C_k| \downarrow$ . Moreover, it belongs to the class  $H_p^{(r)}$ ,  $1 < p \leq 2$  (see (2), p. 436), and does not belong to a better class  $f(x) \notin W_p^{(r)}$  (see the example to Theorem 3), i.e. Theorem 5 is not sharpened in this case either.

The case  $p = 1$  has already been considered earlier (see the example to Theorem 4).

Moscow State University  
named after M. V. Lomonosov

Received  
23 VI 1961

## References

1. S. M. Nikol'skii, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **38**, 44 (1951).
2. A. A. Konjushkov, *Izv. AN SSSR, ser. matem.*, **21**, No. 3, 423 (1957).
3. G. G. Lorentz, *Math. Zs.*, **51**, No. 2 (1948).
4. A. Zygmund, *Trigonometric Series*, Moscow-Leningrad, 1933.
5. M. K. Potapov, *Uch. zap. Ivanovsk. gos. ped. inst.*, **18** (1958).

\* In <sup>(4)</sup> it is proved that  $f^{(\bar{r})}(x) \in H_\infty^\alpha$  only for  $0 < \alpha < 1$ ; however, for  $\alpha = 1$  the proof is analogous.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*