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Abstract

Full Text

MATHEMATICS

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ON THE GEOMETRY OF AN n -DIMENSIONAL SURFACE OF A $(2n - 1)$ -DIMENSIONAL PROJECTIVE SPACE

(Presented by Academician P. S. Aleksandrov, 8 VII 1960)

1. In the present paper a number of basic differential-geometric concepts connected with an n -dimensional surface of a $(2n - 1)$ -dimensional projective space are constructed; among them are generalizations of the concepts of Green's edges and Darboux's pencil of surfaces in the classical projective-differential geometry of a surface. The construction is carried out by the method of canonization of a moving frame. Usually such a canonization consists of separate stages, at each of which the possibility is proved of assigning a definite numerical value to only one of the components of the fundamental object of the manifold ⁽¹⁾ by means of a suitable choice of one of the secondary parameters ⁽²⁾. The lemma formulated below gives a generalization of this method of frame canonization, considerably simplifying it. This lemma applies to any manifolds immersed in the representation space of a finite or infinite Lie group ⁽⁴⁾.

Lemma. Let us assign some constant values X_0^I to the relative components X^I of a geometric object whose field is defined on the immersed manifold by a system of differential equations ⁽³⁾:

$$dX^I = \xi_\alpha^I(X)\omega^\alpha + X_k^I\omega^k, \quad (1)$$

where ω^k and ω^α are, respectively, the principal and secondary forms.

If, from system (1), for the chosen values of the components of the object X_0^I , there is singled out a subsystem of equations linearly independent with respect to the secondary forms ω^α , then there exists a canonical moving frame of the manifold in which, on the whole manifold, those fixed values are preserved by the components whose differentials enter into the indicated subsystem.

2. In a projective space of dimension $N = 2n - 1$ let us consider an n -dimensional surface (M) . With the current point M of the surface we associate a frame consisting of $2n$ analytic points M_I ($I = 0, 1, \dots, 2n - 1$), identifying the point M_0 of the frame with the point M and placing the

points M_i ($i = 1, \dots, n$) in the tangent plane. In such a frame the system of differential equations of the surface has the form

$$\omega^\alpha = 0 \quad (\alpha = n + 1, \dots, 2n - 1). \quad (2)$$

The successive prolongations of system (2) lead to the following differential equations for the components of the fundamental objects:

$$d\Lambda_{ij}^\alpha = \Lambda_{ij\omega}^\alpha \omega^l + \Lambda_{il}^\alpha \omega_j^l - \Lambda_{ij}^\beta \omega_\beta^\alpha - \Lambda_{ij}^\alpha \omega_0^0 + \Lambda_{ijl}^\alpha \omega^l; \quad (3)$$

$$d\Lambda_{ijk}^\alpha = \Lambda_{ljk}^\alpha \omega_i^l + \Lambda_{ilk}^\alpha \omega_j^l + \Lambda_{ijl}^\alpha \omega_k^l - \Lambda_{ijk}^\beta \omega_\beta^\alpha - 2\Lambda_{ijk}^\alpha \omega_0^0 - \\ - \Lambda_{(ij}^\alpha \omega_{k)}^0 + \Lambda_{l(i}^\alpha \Lambda_{jk)}^\beta \omega_\beta^l + \Lambda_{ijkl}^\alpha \omega^l; \quad (4)$$

.....

We shall construct the canonical frame of this surface. Let us specify the components of the fundamental object of second order so that:

$$\Lambda_{ii}^\alpha = 0; \quad (5a)$$

$$\Lambda_{ij}^\alpha \neq 0 \quad (i \neq j); \quad (5b)$$

$$\Delta_i = \begin{vmatrix} \Lambda_{i1}^{n+1} & \dots & \Lambda_{ii-1}^{n+1} & \Lambda_{ii+1}^{n+1} & \dots & \Lambda_{in}^{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Lambda_{i1}^{2n-1} & \dots & \Lambda_{ii-1}^{2n-1} & \Lambda_{ii+1}^{2n-1} & \dots & \Lambda_{in}^{2n-1} \end{vmatrix} \neq 0. \quad (5c)$$

In this case, from system (3) there is singled out a subsystem of equations linearly independent with respect to all secondary forms ω_i^j ($j \neq i$):

$$d\Lambda_{ii}^\alpha = 2\Lambda_{ik}^\alpha \omega_i^k + \Lambda_{iik}^\alpha \omega^k = 0 \quad (\hat{k} \neq i). \quad (6)$$

Consequently, on the basis of the lemma of § 1, there exists a canonical frame in which, on the whole surface (M), the relations (5a) hold. Under such a canonization the secondary forms ω_i^j ($i \neq j$) become linear combinations of the principal forms, which are determined by the relations (6):

$$\omega_i^j = d_{ik}^j \omega^k \quad (i \neq j), \quad (7)$$

where

$$d_{ik}^j = -\frac{1}{2\Delta_i} \Lambda_{iik}^\alpha \delta_{i\alpha}^j,$$

and $\delta_{i\alpha}^j$ is the algebraic complement of the element Λ_{ij}^α of the determinant Δ_i , i.e.

$$\Lambda_{ik}^\alpha \delta_{i\beta}^k = \delta_\beta^\alpha \Delta_i.$$

The geometric meaning of the canonization performed is the assignment of the surface (M) to a system of asymptotic lines.

3. In the tangent space of the surface let us define n fields of quadrics, each of which is determined, with respect to a moving frame associated with the point M of the surface, by the system of equations:

$$x^\alpha = 0; \quad C_{(i)\bar{k}\bar{j}} x^{\bar{k}} x^{\bar{j}} = 0 \quad (8)$$

$$(\bar{k}, \bar{j} = 0, 1, \dots, n; \alpha = n + 1, \dots, 2n - 1; (i) = 1, \dots, n).$$

The system of differential equations of such a field (¹) has the form

$$dC_{(i)\bar{k}\bar{j}} = C_{(i)\bar{l}\bar{j}} \bar{\omega}_{\bar{k}}^{\bar{l}} + C_{(i)\bar{k}\bar{l}} \bar{\omega}_{\bar{j}}^{\bar{l}} + C_{(i)\bar{k}\bar{j}} \theta_{(i)} + C_{(i)\bar{k}\bar{j}\bar{l}} \omega^{\bar{l}}. \quad (9)$$

Let us require that the quadric with number (i) ($i = 1, \dots, n$) pass through the point M of the surface, have contact of second order with the surface in the direction $\omega^i = 0$, and contact of third order along each asymptotic line $\omega^{\hat{j}}$ ($\hat{j} \neq i$). By virtue of these requirements, with a suitable normalization of the coefficients $C_{(i)\bar{k}\bar{j}}$, equations (8) take the form

$$C_{(i)ii} x^i x^i + 2x^0 x^i + 2C_{(i)\hat{k}i} x^{\hat{k}} x^i + C_{(i)\hat{k}\hat{j}} x^{\hat{k}} x^{\hat{j}} = 0; \quad x^\alpha = 0 \quad (\hat{k}, \hat{j} \neq i), \quad (10)$$

where $C_{(i)\hat{k}\hat{j}}$ and $C_{(i)\hat{k}i}$ are determined as certain functions of the fundamental object of third and fourth orders, respectively. Only the coefficient $C_{(i)ii}$ remains arbitrary.

Thus, to each asymptotic line ω^i of the surface (M) in the tangent plane there is invariantly adjoined a one-parameter pencil of quadrics (10), passing through the point M of the surface and having contact of second order with the surface in the direction $\omega^i = 0$, and contact of third order along each asymptotic line $\omega^{\hat{j}}$ ($\hat{j} \neq i$).

4. The equations defining, in the tangent plane, the pole of the tangent to the asymptotic line ω^i with respect to the pencil of quadrics (10) with number (i) , have the following form:

$$x^\alpha = 0; \quad x^0 + C_{\hat{k}\hat{k}}^{(i)} x^{\hat{k}} = 0; \quad x^i = 0. \quad (11)$$

Consequently, the pole is an $(n - 2)$ -dimensional plane, which does not depend on the choice of the parameter $C_{ii}^{(i)}$ of the pencil (10).

The plane (11) with number (i) intersects the tangents to the asymptotic lines $\omega^{\hat{j}}$ ($\hat{j} \neq i$) at the points

$$P_{i\hat{j}} = C_{i\hat{j}}^{(i)} M_0 - M_{\hat{j}}.$$

The n points $P_i = \sum_{\hat{j}} P_{i\hat{j}}$ turn out to be invariantly attached to the surface.

As a result, to each point M of the surface there is invariantly attached an $(n - 1)$ -dimensional plane determined by the points P_i . This plane lies in the tangent plane and does not pass through the point M . It is determined by the system of equations:

$$x^\alpha = 0; \quad \begin{vmatrix} x^0 & x^1 & x^2 & \dots & x^n \\ a_1 & 0 & -1 & \dots & -1 \\ a_2 & -1 & 0 & \dots & -1 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ a_n & -1 & -1 & \dots & 0 \end{vmatrix} = 0, \quad (12)$$

where

$$a_i = \sum_{\hat{j}}^{(i)} C_{i\hat{j}}.$$

We shall call this plane the **second Griin plane**. For $n = 2$ the plane (12) becomes the second Griin edge ([unclear: text in parentheses]).

5. Let us consider the system of quantities

$$K^i = \prod_{\hat{j}} a_{\hat{j}\hat{j}}^i \quad (\hat{j} \neq i),$$

which satisfies the following system of differential equations:

$$dK^i = K^i \left\{ 2 \left[\sum_{\hat{j}} \omega_{\hat{j}}^{\hat{j}} - (n - 1)\omega_0^0 \right] - (n - 1)(\omega_i^i - \omega_0^0) \right\} + K^i \omega'.$$

On the basis of the lemma we canonize the frame so that $K^i = 1$. In this case the forms $(\omega_i^i - \omega_0^0)$ become linear combinations of the principal forms. Next, relying on the lemma, we canonize the frame so that

$$\sum_{\hat{j}}^{(i)} C_{i\hat{j}} = a_i = 0. \quad (13)$$

Then the forms ω_i^0 become linear combinations of the principal forms.

The geometric meaning of such a canonization is as follows. In view of relations (13), the system (12) takes the form:

$$x^\alpha = 0; \quad x^0 = 0.$$

Consequently, the coordinate plane (M_i) coincides with the second Griin plane.

6. Let us attach to the current point of the surface under consideration (M) a surface which is determined by a system of equations of the form

$$A_{IK}^\alpha x^I x^K = 0 \quad (\alpha = n + 1, \dots, 2n - 1; I, K = 0, 1, \dots, 2n - 1) \quad (14)$$

and which contains in the principal part two infinitely close tangents to the asymptotics of one (each) family when the point of tangency is displaced along any other asymptotic. The coefficients A_{ij}^α and $A_{i\beta}^\alpha$ of such a surface

are expressed through the components of the fundamental objects of the second and third orders, respectively.

Transforming equations (14) so that $A_{0\beta}^\alpha = \delta_\beta^\alpha$, we obtain the system of equations of a pencil of surfaces:

$$\Lambda_{ij}^\alpha x^i x^j - 2x^0 x^\alpha - 2A_{i\beta}^\alpha x^i x^\beta - A_{\gamma\beta}^\alpha x^\gamma x^\beta = 0. \quad (15)$$

7. By the **pencil of Darboux hyperquadrics** associated with the current point of the surface (M) we shall mean the pencil of hyperquadrics which is defined by the equation

$$N_\alpha A_{IK}^\alpha x^I x^K = 0, \quad (16)$$

where

$$N_\alpha = \sum_{i, \hat{i}} \delta_{i\alpha}^{\hat{i}} (K^1 \dots K^n)^{\frac{4}{n^2-1}} (K^i K^{\hat{i}})^{-\frac{1}{n+1}} \Delta_i^{-1} \quad (i \neq \hat{i}).$$

For $n = 2$ the pencil of Darboux hyperquadrics (16) becomes the pencil of Darboux surfaces.

8. By the **first Green plane** we shall mean the $(n - 1)$ -dimensional plane that is the polar of the second Green plane with respect to the pencil of Darboux hyperquadrics (16). This plane (normal) has one common point with the osculating plane—the point M —and defines an invariant frame of the surface. For $n = 2$ this plane becomes the first Green edge ⁽³⁾.

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Note: Figure translations are in progress. See original paper for figures.

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