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Abstract

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MATHEMATICS

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ON THE ORDER OF BEST APPROXIMATION OF FUNCTIONS DEFINED ON A CONTINUUM BY LINEAR AGGREGATES OF ELEMENTS OF AN EXTENDABLE BASIS

(Presented by Academician A. N. Kolmogorov on 18 III 1961)

Let K be a bounded continuum in the plane Z, nondegenerate and not separating the plane; let G be a simply connected domain containing K; and let $\mathfrak{A}(G)$ and $\mathfrak{A}(K)$ be the spaces of functions analytic, respectively, in the domain G and on K. Suppose that the function $w = \varphi(z)$ effects a conformal mapping of the doubly connected domain G/K onto the circular annulus

$$1 < |w| < R.$$

V. D. Erokhin, generalizing the well-known theorem of S. N. Bernstein on the order of best approximation by polynomials of functions defined on a continuum ([1]), showed that for every continuum K and any domain G containing it there exists a sequence of analytic functions

$$f_0(z),\ f_1(z), \dots,\ f_j(z), \dots; \qquad f_j(z) \in \mathfrak{A}(G), \quad j = 0, 1, \dots, \tag{1}$$

which has the following property.

Property B. A function $f(z)$, defined on the continuum K, admits, for every $\delta > 0$ and all natural n, approximations by polynomials

$$\sum_{j=0}^{n} c_j^{(n)} f_j(z)$$

of the form

$$\left| f(z) - \sum_{j=0}^{n} c_j^{(n)} f_j(z) \right| \le C(\delta) \left(\frac{1+\delta}{\rho} \right)^n \qquad (1 < \rho < R;\ z \in K),$$

if and only if it is regular in the domain G_ρ, bounded by the preimage of the circle $|w| = \rho$ under the transformation $w = \varphi(z)$ ([2]).

Such a sequence, according to ([2]), is the so-called principal basis of the space $\mathfrak{A}(G)$, which in the special case when the function $\varphi(z)$ maps the exterior of K onto the exterior of a disk ($\varphi(\infty) = \infty$), coincides with the sequence of Faber polynomials.

In the present note it is shown that Property B is possessed by every basis of the space $\mathfrak{A}(G)$ that is extendable to the continuum (i.e., every common basis of the spaces $\mathfrak{A}(G)$ and $\mathfrak{A}(K)$), under the condition that the elements of this basis are numbered in a certain order. The author considers it necessary to emphasize here that the following proof relies essentially on the theorem of V. D. Erokhin stated above.

Let $\{e_n(z)\}_{n=0}^{\infty}$ be a principal basis, and let the sequence (1) be an arbitrary basis of the space $\mathfrak{A}(G)$ extendable to K, and let

$$f_n(z) = \sum_{i=0}^{\infty} a_{in} e_i(z), \qquad e_n(z) = \sum_{i=0}^{\infty} b_{in} f_i(z) \quad (n = 0, 1, \ldots).$$

Changing, if necessary, the numbering of the functions $f_n(z)$, and also multiplying them by suitable constants, we bring the basis to canonical form, ana-

analogously to how this is done in paper ([3]). In doing so we shall keep the previous notation for the functions and coefficients.

Lemma 1. The basis (1) satisfies the conditions:

a)

$$\lim_{j\to\infty} \left[\max_{z\in \overline{G}_\rho} |f_j(z)| \right]^{1/j} = \rho \qquad (1 < \rho < R);$$

b)

$$|a_{ij}| R^{i-j} \le \alpha_j; \qquad |b_{ij}| R^{i-j} \le \alpha_i; \qquad |a_{ij}| \le \alpha_j; \qquad |b_{ij}| \le \alpha_j \quad (i, j = 0, 1, \ldots),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ is a certain nondecreasing sequence of positive numbers such that

$$\lim_{n\to\infty} \alpha_n^{1/n} = 1.$$

The first assertion is proved in paper ([3]), while the second is equivalent to it, as follows from the matrix criterion for a quasi-power basis established by M. G. Khaplanov ([4]).

Lemma 2. For every number $p > 1$ there exists a positive number q such that

$$\sum_{i\ge pn+q} \sum_{j=0}^{n} |a_{ij}|\, |b_{ki}| < \frac{1}{2} \qquad (k = 0, 1, \ldots, n;\ n = 0, 1, \ldots).$$

For the proof it suffices to determine the number $q > 1$ from the condition

$$\frac{1}{2}R^{q/p}(1 - R^{-1/p}) > \sup_{0 \le i < \infty}[\alpha_i^2 R^{(1/p-1)i}]$$

and to use the preceding lemma.

We next consider arbitrary polynomials composed of elements of the basis (1),

$$P_n(z) = \sum_{j=0}^{n} c_j^{(n)} f_j(z); \qquad c_n^{(n)} \ne 0 \quad (n = 0, 1, ...).$$

Let $M_n(K)$ and $M_n(\rho)$ denote, respectively, the maxima of the modulus of $P_n(z)$ on the continuum K and in the domain $\overline{G}_\rho$ $(1 < \rho < R)$.

Lemma 3. The polynomials $P_n(z)$ satisfy the condition

$$\overline{\lim}_{n\to\infty}\left[\frac{M_n(\rho)}{M_n(K)}\right]^{1/n} \le \rho \qquad (1 < \rho < R).$$

Proof. Expand each polynomial $P_n(z)$ in the fundamental basis:

$$P_n(z) = \sum_{j=0}^{n} c_j^{(n)} \sum_{i=0}^{\infty} a_{ij} e_i(z) = \sum_{i=0}^{\infty}\left(\sum_{i=0}^{n} c_j^{(n)} a_{ij}\right) e(z) = \sum_{i=0}^{\infty} \tilde{c}_i^{(n)} e_i(z).$$

From the elementary properties of the fundamental basis established in paper ([2]) it follows that

$$|\tilde{c}_i^{(n)}| \le M_n(K)$$

for all $i,\ n = 0, 1, ...$ We use the identity

$$c_{k^*}^{(n)} = \sum_{j=0}^{n} c_j^{(n)} \sum_{i=0}^{\infty} a_{ij} e_{k^* i} = \sum_{i=0}^{\infty}\sum_{j=0}^{n} c_j^{(n)} a_{ij} b_{k^* i} \qquad (n = 0, 1, ...),$$

in which k^* denotes the ordinal number of any one of the coefficients $c_k^{(n)}$ $(k = 0, 1, ..., n)$ having the largest modulus. With the aid of Lemma 2, for a suitable choice of the numbers p, q, we obtain the inequality

$$|c_{k^*}^{(n)}| \le \sum_{i < pn+q+1}\left|\sum_{j=0}^{n} c_j^{(n)} a_{ij}\right| |b_{k^* i}| + \sum_{i \ge n+q}\sum_{j=0}^{n} |c_j^{(n)}|\,|a_{ij}|\,|b_{k^* i}| \le$$

$$\le \sum_{i < pn+q+1} |\tilde{c}_i^{(n)}|\,|b_{k^* i}| + |c_{k^*}^{(n)}| \sum_{i \ge pn+q}\sum_{j=0}^{n} |a_{ij}|\,|b_{k^* i}| \le$$

$$\le M_n(K)(pn + q + 1)\alpha_{pn+q+1} + \frac{1}{2}|c_{k^*}^{(n)}|.$$

or

$$|c_{k^*}^{(n)}| \leq 2M_n(K)(pn+q+1)\alpha_{pn+q+1} \qquad (n = 0, 1, \ldots).$$

Let $\varepsilon > 0$ be arbitrary. From Lemma 1 it follows that the maximum of the modulus of $f_n(z)$ in the closed domain $\overline{G}_\rho$ does not exceed $C(\varepsilon)(\rho+\varepsilon)^n$. But then, obviously,

$$M_n(\rho) \leq (n+1)|c_{k^*}^{(n)}|C(\varepsilon)(\rho+\varepsilon)^n,$$

whence it follows that

$$\left[\frac{M_n(\rho)}{M_n(K)}\right]^{1/n} \leq \left[2(n+1)(pn+q+1)\alpha_{pn+q+1}C(\varepsilon)\right]^{1/n}(\rho+\varepsilon),$$

and the assertion of the lemma is obtained by passing successively to the limit first as $n \to \infty$ and then as $\varepsilon \to 0$.

Theorem. Every basis of the space $\mathfrak{A}(G)$ extendable to the continuum, after being reduced to canonical form, possesses property with respect to the given continuum.

The proof is based on Lemmas 1a) and 3 and almost word for word repeats the arguments given in ([5]) in the proof of S. N. Bernstein' s theorem.

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REFERENCES

1. S. N. Bernstein, *Communications of the Kharkov Mathematical Society*, ser. 2, **13** (1912).
2. V. D. Erokhin, DAN, **120**, No. 4 (1958).
3. M. M. Dzhrbashyan, *Mathematical Collection*, **59** (95), issue 2 (1961).
4. M. G. Khlaponin, DAN, **80**, No. 2 (1951).
5. A. I. Markushevich, *Theory of Analytic Functions*, 1950.

Note: Figure translations are in progress. See original paper for figures.