L. A. AIZENBERG

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Abstract

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L. A. AIZENBERG

INTEGRAL REPRESENTATIONS OF FUNCTIONS HOLOMORPHIC IN MULTICIRCULAR DOMAINS

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For simplicity of notation we shall use the following notation: $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n)$ (z_j, ζ_j are complex numbers); $\bar{z} = (\bar{z}_1, \dots, \bar{z}_n)$; $|z| = (|z_1|, \dots, |z_n|)$; $k = (k_1, \dots, k_n)$ (k_j are nonnegative integers); $\|k\| = k_1 + \dots + k_n$; $z^k = z_1^{k_1} \cdots z_n^{k_n}$; $z^{-k} = z_1^{-k_1} \cdots z_n^{-k_n}$; $|z|^k = |z_1|^{k_1} \cdots |z_n|^{k_n}$; $\varphi(|\zeta|) = (\varphi_1(|\zeta|), \dots, \varphi_n(|\zeta|))$ ($\varphi_j(|\zeta|)$ are complex-valued functions of $|\zeta|$); $\varphi^k(|\zeta|) = \varphi_1^{k_1} \cdots \varphi_n^{k_n}$; $d\zeta/\zeta = d\zeta_1/\zeta_1 \cdots d\zeta_n/\zeta_n$; $\theta = (\theta_1, \dots, \theta_n)$ ($0 \le \theta_j \le 2\pi$); $d\theta = d\theta_1 \cdots d\theta_n$.

Let D be a complete bounded multicircular (1,2) domain with center at the origin of the space of n complex variables C^n. To each point z of the domain D we associate the point $|z|$ in the positive octant R_+^n of the space R^n; the image of the domain D under this mapping will be denoted by D'. The boundary Γ of the domain D consists of the sets $\Delta_{|\zeta|} = \{z : \ z_j = |\zeta_j| e^{i\theta_j}, \ 0 \le \theta_j \le 2\pi, \ j = 1, \dots, n\}$, where $|\zeta|$ runs over the boundary γ of the domain D'.

1. On the subsets of the set γ we define a finite complex measure $\nu(|\zeta|)$ ((3), p. 121). Taking into account that

$$\frac{1}{(2\pi i)^n} \int_{\Delta_{|\zeta|}} \zeta^k \zeta^{-\tilde{k}} \frac{d\zeta}{\zeta} = |\zeta|^k |\zeta|^{-\tilde{k}} \delta_{k,\tilde{k}}, \tag{1}$$

where $\delta_{k,\tilde{k}} = 0$ if $k \ne \tilde{k}$, $\delta_{k,\tilde{k}} = 1$ if $k = \tilde{k}$, and specifying a sequence $\{\psi_k(|\zeta|)\}$ of measurable and essentially bounded ((3), p. 89) with respect to $|\nu|(|\zeta|)$ ((3), p. 125) functions, we obtain the biorthonormal system on Γ $\{\zeta^k, \ \zeta^k \psi_k(|\zeta|) |\zeta|^{-2k} b_k\}$, where $b_k = \left[\int_\gamma \psi_k(|\zeta|)\, d\nu(|\zeta|)\right]^{-1}$, i.e.

$$\frac{1}{(2\pi i)^n} \int_\gamma d\nu(|\zeta|) \int_{\Delta_{|\zeta|}} \zeta^k \zeta^{-\tilde{k}} \frac{\overline{\psi_{\tilde{k}}(|\zeta|)}}{|\zeta|^{2\tilde{k}}} \overline{b_{\tilde{k}}} \frac{d\zeta}{\zeta} = \delta_{k,\tilde{k}}.$$

Consider the function

$$H(z^k\bar{\zeta}^k) = \sum_k z^k \bar{\zeta}^k \frac{\overline{\psi_k(|\zeta|)}}{|\zeta|^{2k}}\, \bar{b}_k. \tag{2}$$

From the results of (^{2}) the following proposition follows: in order that the series (2), for fixed ζ, converge in the domain D, it is necessary and sufficient that

$$\varlimsup_{\|k\|\to\infty} \left[\frac{|\psi_k(|\zeta|)|}{|\zeta|^k}\, |b_k|\, d_k(D)\right]^{1/\|k\|} \leq 1, \quad \text{where } d_k(D) = \sup_{z\in D} |z|^k = \sup_{|z|\in D'} |z|^k. \tag{3}$$

Suppose that inequality (3) is satisfied for almost all (in the sense of the measure $|\nu|(|\zeta|)$) $|\zeta|$ from γ, and that for each fixed z, $z \in D$, the series (2) converges in the mean (with respect to the product of the measure $|\nu|(|\zeta|)$ and the measure corresponding to the differential $d\theta$). Then the following theorem is valid:

Theorem 1. Let $f(z)$ be holomorphic in the domain D and continuous in the closed domain $\overline{D}$. For each point z, $z \in D$,

$$f(z) = \frac{1}{(2\pi i)^n} \int_\gamma d\nu(|\xi|) \int_{\Delta|\xi|} f(\zeta)\, H(z,\zeta)\, \frac{d\zeta}{\zeta}. \tag{4}$$

Proof. The validity of formula (4) for functions holomorphic in the closed domain $\overline{D}$ is easily verified by expanding $f(\zeta)$ on the right-hand side of (4) into a multiple power series. Then, by means of the usual arguments, one passes to functions holomorphic in the domain D and continuous in the closed domain $\overline{D}$. We note that the kernel $H(z,\zeta)$ in the integral representation (4) is holomorphic in z.

2. Put $\psi_k(|\xi|) = \varphi^k(|\xi|)$. Then

$$H(z,\zeta) = \sum_k z^k \bar{\zeta}^k \varphi^{-k}(|\xi|)|\xi|^{-2k}\, \bar{b}_k$$

$$= h(z_1\bar{\zeta}_1\overline{\varphi_1(|\xi|)}|\xi_1|^{-2}, \dots, z_n\bar{\zeta}_n\overline{\varphi_n(|\xi|)}|\xi_n|^{-2}),$$

where h is a holomorphic function of n complex variables.

Theorem 2. Let a holomorphic function be given

$$h(z) = \sum_k a_k z^k. \tag{5}$$

In order that the function h can be used as a kernel in an integral representation of the form

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\gamma} d\nu(|\xi|) \int_{\Delta|\xi|} f(\zeta)\, h\left(z_1 \bar{\zeta}_1 \frac{\overline{\varphi_1(|\xi|)}}{|\xi_1|^2}, \dots, z_n \bar{\zeta}_n \frac{\overline{\varphi_n(|\xi|)}}{|\xi_n|^2} \right) \frac{d\zeta}{\zeta}, \tag{6}$$

it is sufficient that the following three conditions be satisfied (conditions 1) and 2) are also necessary):

1) There exists a finite complex measure $\nu(|\xi|)$, defined on subsets of the set γ, such that the moment problem

$$\frac{1}{a_k} = \int_{\gamma} \overline{\varphi^k(|\xi|)}\, d\nu(|\xi|) \tag{7}$$

is solvable in the class of functions $\varphi(|\xi|) = (\varphi_1(|\xi|), \dots, \varphi_n(|\xi|))$ that are measurable and essentially bounded with respect to $|\nu|(|\xi|)$.

2) For almost all $|\xi|$, $|\xi| \in \gamma$ (in the sense of the measure $|\nu|(|\xi|)$),

$$\overline{\lim_{\|k\|\to\infty}} \left[|\varphi^k(|\xi|)|\, |\xi|^{-k}\, |a_k|\, d_k(D) \right]^{1/\|k\|} \leq 1.$$

3)

$$\overline{\lim_{\|k\|\to\infty}} \left[|a_k|\, d_k(D) \int_{\gamma} \frac{|\varphi^k(|\xi|)|}{|\xi|^k}\, d|\nu|(|\xi|) \right]^{1/\|k\|} \leq 1.$$

We note that if the function h can be used as a kernel in the integral representation (6), then all a_k are nonzero. Therefore the moment problem (7) always has meaning.

To obtain more transparent conditions for the existence of the integral representation (6), consider the case where $\varphi_j(|\xi|) \geq 0$, $j = 1, \dots, n$.

Theorem 3. Let a holomorphic function h, representable by the series (5), be given. In order that the function h can be used as a kernel in an integral representation of the form (6), where $\varphi_j(|\xi|) \geq 0$, $\nu(|\xi|)$ is nonnegative, it is necessary and sufficient that the following two conditions be satisfied:

1) There exists a finite nonnegative measure $\nu(|\xi|)$, defined on subsets of the set γ, whose support is γ, such that the moment problem (7) is solvable in the class of nonnegative functions measurable and essentially bounded with respect to the measure $\nu(|\xi|)$.

2) If $\varphi(|\xi|)$ is a solution of the moment problem (7), then for all k

$$\sup_{\gamma} \operatorname{vrai} \varphi^k(|\xi|)\, [\max_{\gamma} |\xi|^k]^{-1} = \sup_{\gamma} \operatorname{vrai} \varphi^k(|\xi|)|\xi|^{-k} \; * \, . \tag{8}$$

* sup vrai is considered with respect to the measure $\nu(|\xi|)$.

Proof. For any nonnegative essentially bounded function $F(|\xi|)$,

$$\lim_{m\to\infty} \left[\int_{\gamma} F^m(|\xi|)\, d\nu(|\xi|) \right]^{1/m} = \sup_{\gamma} \operatorname{vrai} F(|\xi|),$$

and therefore the inequality

$$\sup_{\gamma} \operatorname{vrai}\, \varphi^k(|\xi|) \cdot \left[\max_{\gamma} |\xi|^k \right]^{-1} \geq \sup_{\gamma} \operatorname{vrai}\, \varphi^k(|\xi|)|\xi|^{-k} \tag{9}$$

is equivalent to conditions 2) and 3) of Theorem 2. The inequality inverse to (9) is obvious.

3. Let us indicate an important application of the results of § 2. Fix a domain D_0 and put $\varphi_j(|\xi|) = |\xi_j|^2$, $j = 1, \dots, n$. Consider the function $h_0(z) = \sum_k c_k z^k$, where

$$\frac{1}{c_k} = \int_{\gamma_0} |\xi|^{2k}\, d\nu_0(|\xi|). \tag{10}$$

By Theorem 3, the integral representation

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\gamma_0} d\nu_0(|\xi|) \int_{\Delta|\xi|} f(\zeta)\, h_0(z_1\overline{\xi}_1, \dots, z_n\overline{\xi}_n)\, \frac{d\zeta}{\zeta}, \tag{11}$$

holds, where γ_0 is the boundary of the domain D_0' in R^n_+. The kernel in the integral representation (11) is a function holomorphic in z and in $\overline{\zeta}$—the so-called Szegő kernel. It is completely determined by specifying the domain D_0 and the measure $\nu_0(|\xi|)$. Each domain has its own Szegő kernel.

Let us consider an arbitrary domain D and pose the following problem: can one choose functions $\varphi_j(|\xi|)$, $j = 1, \dots, n$, and a measure $\nu(|\xi|)$ so that, for the domain D, the integral representation (6) holds, where $h(z) \equiv h_0(z)$.

It is always possible to choose $\varphi_j(|\xi|)$ and $\nu(|\xi|)$ so that the a_k from (7) are equal, respectively, to the c_k from (10). If the $\varphi_j(|\xi|)$ chosen in this way satisfy condition (8), then the required integral representation exists. Condition (8) will be satisfied if D belongs to a certain class of domains depending on the domain D_0. Thus, if a holomorphic function h_0 is the Szegő kernel in the integral representation (11) for the domain D_0, then it can be used as the kernel in the integral representation (6) for a certain class of domains D. We give several examples, considering for simplicity the case of two complex variables.

A. For the hypersphere $D_0 = \{z:\ |z_1|^2 + |z_2|^2 < 1\}$ the integral representation

$$f(z) = \frac{2}{(2\pi i)^2} \int_0^1 |\xi_1|\, d|\xi_1| \int_{\Delta|\xi|} \frac{f(\zeta)}{(1 - z_1\bar{\xi}_1 - z_2\bar{\xi}_2)^2}\, \frac{d\zeta}{\zeta}. \tag{12}$$

holds. If the domain $D = \{z:\ |z_2| < \Phi(|z_1|)\}$ is convex and the function Φ' is absolutely continuous, then for this domain we obtain the Temlyakov integral (4,5)

$$f(z) = \frac{1}{(2\pi i)^2} \int_0^r \varphi_1'(|\xi_1|)\, d|\xi_1| \times$$

$$\times \int_{\Delta|\xi|} \frac{f(\zeta)}{\left(1 - z_1\bar{\xi}_1\varphi_1(|\xi_1|)|\xi_1|^{-2} - z_2\bar{\xi}_2\varphi_2(|\xi_2|)|\xi_2|^{-2}\right)^2}\, \frac{d\zeta}{\zeta},$$

where

$$\varphi_1(|\xi_1|) = |\xi_1|\Phi'(|\xi_1|)\left[|\xi_1|\Phi'(|\xi_1|) - \Phi(|\xi_1|)\right]^{-1}, \qquad \varphi_2(|\xi_2|) = 1 - \varphi_1(|\xi_1|),$$

$$|\xi_2| = \Phi(|\xi_1|), \qquad 0 \le |\xi_1| \le r.$$

In exactly the same way, Theorem 3 makes it possible to obtain integral representations that are extensions to the case of n complex variables of Temlyakov' s integral representations (6,7). We note that Temlyakov' s integral representation is the only one among the integral representations known so far that can be reduced to the form (6) (and not (11)). It served as the impetus for the present work.

B. For the hypercone $D_0 = \{z:\ |z_1| + |z_2| < 1\}$ there holds, as can be shown, the integral representation

$$f(z) = \frac{1}{(2\pi i)^2} \int_0^1 d|\zeta_1| \int_{\Delta_{|\zeta|}} \frac{(1 - x - y)(1 + 2xy - x^2 - y^2) + 8xy}{[1 + x^2 + y^2 - 2x - 2y - 2xy]^2}\, f(\zeta)\frac{d\zeta}{\zeta},$$

$$x = \bar{z}_1\zeta_1, \qquad y = \bar{z}_2\zeta_2.$$

This integral representation can be extended to a class of domains wider than the class of convex domains.

C. The results of §§ 1 and 2 can be extended to certain unbounded domains. For example, for the domain

$$D_0 = \{z:\ |z_2|^2 < 2|\ln|z_1||,\ 0 < |z_1| < 1\}$$

there holds the integral representation

$$f(z) = \frac{2}{(2\pi i)^2}\int_0^1 |\zeta_1|\,d|\zeta_1|\int_{\Delta_{|\zeta|}} \frac{e^{z_2\bar\zeta_2} f(\zeta)}{(1 - z_1\bar\zeta_1 e^{z_2\bar\zeta_2})^2}\frac{d\zeta}{\zeta}.$$

4. Let us generalize the preceding results in the following way. A set E, $E \subset \gamma$, will be called a boundary for monomials if for every k there exists a point $|\xi| \in E$ such that $|\xi|^k = d_k(D)$. Take a finite or countable collection of closed sets γ_p, $\gamma_p \subset \gamma$, such that the set $\bigcup_p \gamma_p$ is a boundary for monomials. Divide the collection of vectors k into sets N_p, assigning the vector k to one of those sets N_p for which

$$\sup_{|\zeta|\in\gamma_p} |\zeta|^k = d_k(D).$$

Let a finite complex measure $\mu_p(\theta)$ be given on $\Delta_{|\zeta|}$, $|\zeta| \in \gamma_p$, and suppose that for $\tilde k \in N_p$ and arbitrary k a property analogous to (1) holds. On the subsets of each set γ_p let us define a finite complex measure $\nu_p(|\zeta|)$. Consider a sequence $\{\psi_k(|\zeta|)\}_{k\in N_p}$, $|\zeta| \in \gamma_p$, of functions measurable and essentially bounded with respect to $|\nu_p|(|\zeta|)$. Introduce the functions

$$H_p(z,\xi) = \sum_{k\in N_p} z^k\bar\xi^k \frac{\overline{\psi_k(|\xi|)}}{|\xi|^{2k}}\, b_k, \tag{13}$$

where

$$b_k = \left[\int_{\gamma_p} \overline{\psi_k(|\xi|)}\, d\nu_p(|\xi|)\right]^{-1}.$$

Suppose that inequality (3) ($k \in N_p$) is satisfied for almost all (in the sense of the measure $|\nu_p|(|\xi|)$) $|\xi|$ from γ_p, and that for each fixed z, $z \in D$, the series (13) converges in the mean (with respect to the product of the measures $|\nu_p|(|\xi|) \times |\mu_p|(\theta)$). Then the following theorem is valid:

Theorem 4. Let $f(z)$ be holomorphic in the domain D and continuous in the closed domain $\overline{D}$. For every point z, $z \in D$,

$$f(z) = \frac{1}{(2\pi i)^n} \sum_p \int_{\gamma_p} d\nu_p(|\xi|) \int_{\Delta_{|\xi|}} f(\xi) H_p(z, \xi)\, d\mu_p(\theta).$$

Theorems 2 and 3 extend analogously. If, in addition, each γ_p consists of a single point and the measures $\mu_p(\theta)$ are chosen in a certain definite way, then Leau' s integral representation (^{8}) is obtained; if the domain D is a Weil domain, then the Weil integral (^{1}) is obtained.

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Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

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Note: Figure translations are in progress. See original paper for figures.