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# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

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**HOMOGENEOUS FUNCTIONS AND SOME PROBLEMS OF INTEGRAL GEOMETRY IN SPACES OF CONSTANT CURVATURE**

*(Presented by Academician I. G. Petrovskii, 7 VII 1960)*

1. In the present work we consider certain integral transformations of homogeneous generalized functions in Euclidean spaces, as well as the transformations generated by them in spaces of functions defined on spaces of constant curvature. The results obtained make it possible, in particular, to give a solution of the problem of reconstructing a function from its known integrals over all possible totally geodesic surfaces in spaces of constant curvature. For lack of space, only the case of the proper Euclidean and elliptic spaces will be considered in detail.

2. As in <sup>(3)</sup>, we shall consider the space  $\Phi$  of basic functions defined on the  $n$ -dimensional Euclidean space  $R_n$  and possessing the following properties: 1) the functions  $\varphi \in \Phi$  are infinitely differentiable; 2) the functions  $\varphi$ , together with all their derivatives, decrease as

$$r \rightarrow \infty \quad \left( r = \sqrt{x_1^2 + \dots + x_n^2} \right)$$

faster than any power of  $1/r$ ; 3) these functions are orthogonal to all polynomials. In the space  $\Phi'$  of generalized functions over  $\Phi$ , in <sup>(3)</sup> there were defined convolution operators with a generalized function  $R_\lambda$ , where  $\lambda$  is a complex parameter:  $f \rightarrow R_\lambda * f$ , where

$$R_\lambda = \begin{cases} 2^\lambda \pi^{-n/2} \Gamma[(\lambda + n)/2] [\Gamma(-\lambda/2)]^{-1} r^{-\lambda-n}, & \text{for } \lambda = -n, -n - 2; \\ \frac{(-1)^k}{2^{n+2k-1} \pi^{n/2} \Gamma(k + n/2) k!} r^{2k} \ln r, & \text{for } \lambda = -n - 2k; k = 0, 1. \end{cases} \quad (1)$$

Here we shall consider the action of the operators  $R_\lambda *$  on the subspace  $\Phi'_0$  of homogeneous generalized functions of the space  $\Phi'$ . It is important that the group of operators  $R_\lambda *$  carries the space  $\Phi'_0$  into itself; more precisely, if  $f$  is a homogeneous generalized function of the space  $\Phi'$  of degree  $\mu$ , then  $R_\lambda * f$  is a homogeneous generalized function from  $\Phi'$  of degree  $\mu - \lambda$ .

Let us consider in somewhat greater detail the structure of the space  $\Phi'_0$ . The space  $\Phi$  is a subspace of the space  $S$  of infinitely differentiable functions decreasing at infinity faster than any power of  $1/r$ . At the same time, every generalized function from the space  $S'$  of generalized functions over  $S$  is simultaneously a generalized function from  $\Phi'$ , and all functions from  $S'$  that differ from one another by a polynomial define one and the same generalized function from  $\Phi'$ . Thus, a homogeneous function from  $\Phi'$  may be represented by an entire class of functions from  $S'$ . However, each such class, provided only that the degree of homogeneity  $\mu$  is not a natural number or zero, contains exactly one function homogeneous also from the point of view of the space  $S'$ . If, however, the degree of homogeneity  $\mu = k$ , where  $k = 0, 1, \dots$ , then the totality of homogeneous functions from  $S'$  differing

differing by polynomials of degree  $k$ , will determine one and the same homogeneous function from  $\Phi'$ , and moreover some homogeneous functions from  $\Phi'$  for  $\mu = k$  cannot at all be represented by homogeneous functions from  $S'$ , but can be represented by adjoined (see (1)) functions of degree  $k$ , again with accuracy up to a polynomial of the same degree. For example, the homogeneous function  $R_{-n-2k} \in \Phi'_0$  is represented in formula (1) by a function adjoined from the point of view of the space  $S'$ . Thus one may regard the space  $\Phi'_0$  as consisting of all generalized homogeneous functions of degree  $\mu \neq 0, 1, \dots$  of the space  $S'$ , and also of the classes of homogeneous and adjoined functions of degree  $k$  ( $k = 0, 1, \dots$ ) of this space, differing from one another by a polynomial of degree  $k$ .

3. Let  $S_{n-1}$  be the  $(n-1)$ -dimensional elliptic space, represented as the hypersphere of unit radius in  $R_n$  with diametrically opposite points identified. Every function on  $S_{n-1}$  may be regarded as an even function ( $f(\omega) = f(-\omega)$ , where  $\omega_1^2 + \dots + \omega_n^2 = 1$ ) on such a hypersphere. To each complex  $\lambda \neq 0, 2, \dots, -n, -n-2, \dots$  we associate the space  $H_\lambda$  of all generalized functions  $f(\omega)$ , defined on  $S_{n-1}$ ; to each  $\lambda = 2k$  ( $k = 0, 1, \dots$ ) we associate the space  $H_{2k}$  of classes of generalized functions differing from one another by polynomials of degree  $2k$  in  $\omega_i$ , and to each  $\lambda = -n - 2k$  ( $k = 0, 1, \dots$ ) the space  $H_{-n-2k}$  of generalized functions orthogonal to all polynomials of degree  $2k$ . Functions  $f(\omega) \in H_\lambda$  will be denoted by  $f_\lambda(\omega)$ , and we agree to extend them to  $R_n$  in the form  $r^\lambda f_\lambda(\omega)$  (see (1)). Then it can be shown that  $r^\lambda f_\lambda(\omega) \in \Phi'_0$ . We note that the unification of the notation of functions in the spaces  $H_\lambda$  for  $\lambda = 0, 2, \dots$ , and also for  $\lambda = -n, -n-2, \dots$ , is connected with the circumstance that the adjoined, as well as  $\delta$ -like, functions from  $S'$  cannot be represented in the form  $r^\lambda f_\lambda(\omega)$ , where  $f_\lambda(\omega)$  is a function on the hypersphere.

We now subject the function  $r^\lambda f_\lambda(\omega)$  to the transformation  $R_\mu*$ . Then, assuming first that  $\lambda - \mu \neq 0, 2, \dots, -n, -n-2, \dots$ , we shall have

$$R_\mu * r^\lambda f_\lambda(\omega) = r^{\lambda-\mu} g_{\lambda-\mu}(\omega). \quad (2)$$

Thus we obtain a certain transformation  $f_\lambda \rightarrow g_{\lambda-\mu} = Z_{\lambda-\mu}^\lambda f_\lambda$ . The operator  $Z_{\lambda-\mu}^\lambda$  evidently transforms the space  $H_\lambda$  into  $H_{\lambda-\mu}$ . If  $f_\lambda(\omega)$  is an ordinary function restricted to  $S_{n-1}$ , then for  $\operatorname{Re} \mu < 0$  and  $\lambda \neq -n, -n-2, \dots$  the transformation  $f_\lambda \rightarrow g_{\lambda-\mu}$  can be written in integral form

$$g_{\lambda-\mu}(\omega) = \frac{2^\mu \Gamma(p) \Gamma(q)}{\pi^{n/2} \Gamma(-\mu/2)} \int_{S_{n-1}} f_\lambda(\alpha) F\left(p, q; \frac{1}{2}; (\alpha\omega)^2\right) d\alpha, \quad (3)$$

where  $F$  is the hypergeometric function,  $p = (\lambda + n)/2$ ,  $q = (\mu - \lambda)/2$ ,  $(\alpha\omega) = \alpha_1\omega_1 + \dots + \alpha_n\omega_n$  is the cosine of the distance between the points  $\alpha$  and  $\omega$ , and the integral over  $S_{n-1}$  may be understood as one half of the integral over the unit hypersphere in  $R_n$ . For  $\operatorname{Re} \mu \geq 0$  the indicated transformation can also be written in the form (3), understanding this integral in the regularized sense, i.e. as the analytic continuation of this integral from the domain  $\operatorname{Re} \mu < 0$ . Let us note here that for  $\mu = 0$  this transformation is the identity, while for  $\mu = 2k$  the operator  $Z_{\lambda-\mu}^\lambda$  becomes differential:

$$\begin{aligned} Z_{\lambda-2k}^\lambda &= (-1)^k [\Delta + (\lambda - 2k + 2)(\lambda + n - 2k)] \times \\ &\times [\Delta + (\lambda - 2k + 4)(\lambda + n - 2k + 2)] \dots [\Delta + \lambda(\lambda + n - 2)], \end{aligned} \quad (4)$$

where  $\Delta$  is the Laplace operator on  $S_{n-1}$ . We also note that for  $\lambda = -n, -n-2, \dots$  the transformation  $f_\lambda \rightarrow g_{\lambda-\mu}$  can be understood as the limiting value of formula (3) as  $\lambda \rightarrow -n-2k$ , taking into account that for  $\lambda = -n-2k$  the integral in formula (3) vanishes (by virtue of the orthogonality of  $f_{-n-2k}$  to polynomials of degree  $2k$ ), while the factor  $\Gamma(p)$  becomes infinite. Now let  $\lambda - \mu = 2k$  or  $\lambda - \mu = -n - 2k$  ( $k = 0, 1, \dots$ ). Then

formula (2) defines the function  $g_{\lambda-\mu}$  in the same way as before, under the condition that  $f_\lambda(\omega)$  is not a polynomial in  $\omega_i$  of degree  $2k$ . If, however,  $f_\lambda(\omega)$  is such a polynomial, then, considering this case as a limiting one and using (3), we see that in this case it is necessary to set  $g_{\lambda-\mu} = 0$ .

Thus, the operators  $Z_{\lambda-\mu}^\lambda$  are defined for all  $\lambda, \mu$ , and in those cases when they cannot be defined by formula (2), i.e., when  $f_\lambda$  is a polynomial of degree  $2k$  and  $\lambda - \mu$  is equal to  $2k$  or  $-n - 2k$ , by definition we have  $Z_{\lambda-\mu}^\lambda f_\lambda = 0$ .

For  $\lambda - \mu \neq 0, 2, \dots, -n, -n-2, \dots$ , the operators  $Z_{\lambda-\mu}^\lambda$  satisfy two relations:

$$Z_{\lambda-\mu-\nu}^{\lambda-\mu} (Z_{\lambda-\mu}^\lambda f_\lambda) = Z_{\lambda-\mu-\nu}^\lambda f_\lambda; \quad Z_\lambda^{\lambda-\mu} (Z_{\lambda-\mu}^\lambda f_\lambda) = f_\lambda. \quad (5)$$

- Let us now consider the Fourier transform of the generalized function  $r^\lambda f_\lambda$ , where  $f_\lambda \in H_\lambda$ . Then we shall have  $(2\pi)^{-n/2} F[r^\lambda f_\lambda(\omega)] = r^{-\lambda-n} h_{-\lambda-n}(\alpha)$ . Thereby we have defined the transformation  $f_\lambda(\omega) \rightarrow h_{-\lambda-n}(\alpha) = X_{-\lambda-n}^\lambda f$ . We shall interpret the function

$h_{-\lambda-n}(\alpha)$  as a function on the hypersphere of the space  $S_{n-1}$ , polar to the point  $\alpha$ . If  $f_\lambda(\omega)$  is an ordinary function restricted to  $S_{n-1}$ , then the transformation  $f_\lambda \rightarrow h_{-\lambda-n}$  can be written in integral form

$$h_{-\lambda-n}(\alpha) = \frac{2^{\lambda+n/2}\Gamma[(\lambda+n)/2]}{\pi^{(n-1)/2}\Gamma[(-\lambda-n+1)/2]} \int_{S_{n-1}} f_\lambda(\omega) |(\alpha\omega)|^{-\lambda-n} d\omega. \quad (6)$$

This integral converges for  $\text{Re } \lambda < -n + 1$ ; for the remaining  $\lambda$  it must be understood in the sense of analytic continuation. For  $\lambda = -n - 2k$ ;  $k = 0, 1, \dots$ , this formula must be understood in the limiting sense, since the integral in it is a polynomial of degree  $2k$ , i.e., the zero element in the space  $H_{2k}$ , while the factor  $\Gamma[(\lambda+n)/2]$  tends to infinity. For  $\lambda = -n + 1$ , (6) takes the form

$$h_{-1}(\alpha) = (2\pi)^{-n/2+1} \int_{(\alpha\omega)=0} f_{-n+1}(\omega) d\omega_\alpha, \quad (7)$$

where the integral is extended over the hypersphere polar to the point  $\alpha$ .

Denote by  $p_\nu^\lambda$  the operator carrying functions  $f_\lambda$  from the space  $H_\lambda$  into  $H_\nu$  without actually changing them:  $f_\nu = p_\nu^\lambda f_\lambda$  ( $\lambda, \nu \neq 0, 2, \dots, -n, -n-2, \dots$ ). Then the operators  $Z_{-\lambda-\mu}^\lambda$  and  $X_{-\lambda-n}^\lambda$  turn out to be connected by the relation

$$X_{-\lambda-\mu}^{-\lambda+\mu-n} [p_{-\lambda+\mu-n}^{-\lambda-n} (X_{-\lambda-n}^\lambda f_\lambda)] = Z_{-\lambda-\mu}^\lambda f_\lambda. \quad (8)$$

5. Let us now consider the problem of recovering the function  $f(\omega)$  from its known integrals  $h(\alpha)$  over all possible hyperspheres  $(\alpha\omega) = 0$  in  $S_{n-1}$ . Using the inverse Fourier transform for the function  $r^{-1}h_{-1}(\alpha)$ , we may write  $f_{-n+1} = X_{-n+1}^{-1}h_{-1}$ . For odd  $n$ , in view of (6) and (7), this gives

$$f(\omega) = \frac{(-1)^{(n-1)/2}[(n-3)/2]!(n-2)!!}{2^{(n-3)/2}\pi^{n-3/2}} \int_{S_{n-1}} |(\alpha\omega)|^{-n+1} d\alpha \int_{(\alpha\beta)=0} f(\beta) d\beta_\alpha, \quad (9)$$

where the integral is understood in the regularized sense, as was indicated in connection with formula (6). For even  $n$  an explicit writing of the resulting solution is inconvenient. Therefore we shall present it in another form. Put in (8)  $\lambda = -n + 1$ ,  $\mu = -n + 2$ . Then the left-hand side of this equality will be:

$$(2\pi)^{-n+2} \int_{(\alpha\omega)=0} d\alpha_\omega \int_{(\alpha\beta)=0} f_{-n+1}(\beta) d\beta_\alpha = (2\pi)^{-n+2} If(\omega), \quad (10)$$

where  $If(\omega)$  is the integral, over the pencil of hyperspheres passing through the point  $\omega$ , of the integrals of  $f(\beta)$  over these hyperspheres. We obtain  $(2\pi)^{-n+2}If = Z_{-1}^{-n+1}f_{-n+1}$ , whence, in view of (6),  $f_{-n+1} = (2\pi)^{-n+2}Z_{-n+1}^{-1}If$ .

For odd  $n$  this gives a formula equivalent to (9):

$$f(\omega) = \frac{(-1)^{\frac{n-1}{2}} \left(\frac{n-3}{2}\right)!(n-2)!}{\pi^{3/2(n-1)}} \int_{S_{n-1}} I f(\alpha) F\left(\frac{n-1}{2}, \frac{n-1}{2}; \frac{1}{2}; (\alpha\omega)^2\right) d\alpha, \quad (11)$$

where the integral is again understood in the regularized sense. For even  $n$ , using (4), we obtain

$$f(\omega) = (2\pi)^{-n+2} [1(n-3) - \Delta] [3(n-5) - \Delta] \dots [(n-3)1 - \Delta] I f(\omega). \quad (12)$$

The last formula was first obtained by Helgason (2).

6. An analogous theory holds also in non-Euclidean spaces  $S_{n-1}^{(q)}$  of either sign of curvature and of any index  $q$ . In the present case the role of the functionals  $R_\lambda$  is played by the generalized functions  $R_\lambda^t$ , whose explicit form can be obtained from formula (1) if in it  $r$  is replaced by  $(r^2 + i0)^{1/2}$  and the result is multiplied by  $e^{\pi q i/2}$ , where this time

$$r^2 = -x_1^2 - \dots - x_q^2 + x_{q+1}^2 + \dots + x_n^2.$$

It is then expedient to consider the spaces  $S_{n-1}^{(q)}$  of negative and positive curvature together, interpreting them as hyperspheres of imaginary-unit and unit radii with diametrically opposite points identified in the pseudo-Euclidean space  $R_n^{(q)}$  of index  $q$ . Considering in  $R_n^{(q)}$  the generalized functions  $|r|^\lambda f_\lambda(\omega)$ , where  $f_\lambda$  is an even generalized function on the mentioned hyperspheres, we define in the same way as before the operators  $Z_{\lambda-\mu}^\lambda$  and  $X_{\lambda-n}^\lambda$ . Here, if  $f_\lambda$  as  $\omega \rightarrow \infty$  behaves like  $\rho^t$ , where  $\rho$  is the distance from  $\omega$  to a fixed point of  $S_{n-1}^{(q)}$ , and  $t$  does not exceed  $\text{Re}(\lambda + 1)$  and  $\text{Re}(\lambda - \mu + n + 1)$ , then the transform  $g_{\lambda-\mu} = Z_{\lambda-\mu}^\lambda f_\lambda$  can be written with the aid of formula (3), where the kernel  $F(\dots; (\alpha\omega)^2)$  must be replaced by  $F(\dots; (\alpha\omega)^2 - i0)$ ; this time

$$(\alpha\omega) = -\alpha_1\omega_1 - \dots - \alpha_q\omega_q + \alpha_{q+1}\omega_{q+1} + \dots + \alpha_n\omega_n$$

and the integral extends over both spaces  $S_{n-1}^{(q)}$ .

Let now the operator  $p_\nu^\lambda$  take  $f_\lambda$  into  $f_\nu^*$  in such a way that  $f_\nu^* = f_\lambda$  for  $(\omega, \omega) > 0$ , and  $f_\nu^* = e^{(\lambda-\nu)\pi i/2} f_\lambda$  for  $(\omega, \omega) < 0$ . Then, denoting

$$(2\pi)^{-n+2} I f = X_{-1}^{-n+1} [p_{-n+1}^{-1} (X_{-1}^{-n+1} f_{-n+1})],$$

we obtain the formula

$$f_{-n+1} = Z_{-n+1}^{-1} I f,$$

which gives the solution of the problem of reconstructing the function  $f$  from its known integrals over hyperplanes in the space  $S_{n-1}^{(q)}$ . In the case of even  $n$  it takes the form

$$f(\omega) = (2\pi)_*^{-n+2} [\varkappa_1 \cdot (n-3) - \Delta] [\varkappa_3(n-5) - \Delta] \dots [\varkappa(n-3) \cdot 1 - \Delta] I f(\omega), \quad (13)$$

where  $\varkappa$  is the curvature of the space at the point  $\omega$ . If  $n$  is odd, then in this case it is simpler to use the solution of the problem in the form

$$f_{-n+1} = X_{-n+1}^{-1} (X_{-1}^{-n+1} f_{-n+1}).$$

This solution again leads to formula (9), where the outer integration extends over both spaces  $S_{n-1}^{(q)}$ . Let us note here that if one considers Lobachevsky space  $S_{n-1}^{(1)}$  and on it a finite function equal to zero on the space of positive curvature, then in formula (9) the inner integral is a finite function, equal to zero on the space  $S_{n-1}^{(1)}$  of negative curvature, and, consequently, the outer integral extends over the space of positive curvature. Formula (13) for the case of a finite function in Lobachevsky space of negative curvature also belongs to Helgason <sup>(2)</sup>.

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*Note: Figure translations are in progress. See original paper for figures.*

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