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Abstract

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MATHEMATICS

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ON EXACT WEYL MULTIPLIERS FOR UNCONDITIONAL CONVERGENCE

(Presented by Academician A. N. Kolmogorov on 19 VII 1961)

The present note is a continuation of the author' s investigations ([5–7]). As is known (see ([1]), pp. 188-191), D. E. Menshov and Rademacher independently proved that $\log^2 n$ is a Weyl convergence multiplier for any orthonormal system. In addition, D. E. Menshov proved ([2–4]) that there exists an orthonormal system of polynomials, bounded in the aggregate, for which $\log^2 n$ is an exact Weyl multiplier.

Let us note that up to now it is unknown whether every orthonormal system of functions has an exact Weyl multiplier.

Haar constructed a complete orthonormal system of functions $\chi_m(x)$ $(m = 1, 2, \ldots)$, for which he showed that $\omega(m) \equiv 1$ is a Weyl convergence multiplier.

In paper ([5]) we announced for the first time that the sequence $\omega(m) \equiv 1$ is **not** a Weyl multiplier for unconditional convergence of series with respect to the Haar system $\{\chi_m(x)\}$.

The question naturally arises: what sequence $\omega(m)$ is an exact Weyl multiplier for unconditional convergence of series with respect to the Haar system? In the present note an answer to this question will be given.

In paper ([6]) we formulated, and in paper ([7]) (see Theorem 5.8) proved, that the following is true:

Theorem A. *The sequence* $(\log m)^\alpha$ *for* $\alpha > 1$ *is (and for* $\alpha < 1$ *is not) a Weyl multiplier for unconditional convergence almost everywhere of series with respect to the Haar system* $\{\chi_m(x)\}$.

Thus, we see that if the Haar system had an exact Weyl multiplier for unconditional convergence, then it would have the order of a function close to $\log m$.

As far as we know, Theorem A is the first result concerning a classical orthonormal system in which the finality of a Weyl multiplier for unconditional convergence is involved. As for trigonometric series (and also series with respect to the Walsh system), there are no definitive results in the indicated direction.

Further, in paper ([7]) (see Remark 5.7) we established that the following is true:

Theorem B. *If a positive sequence $\omega(m) \uparrow$ is such that*

$$\sum_{m=1}^{\infty} \frac{1}{m\omega(m)} < \infty,$$

then every series

$$\sum_{m=1}^{\infty} a_m \chi_m(x)$$

converges unconditionally almost everywhere on $[0, 1]$, as soon as

$$\sum_{m=1}^{\infty} a_m^2 \omega(m) < \infty.$$

It turns out that Theorem B is final. Namely, the following is true:

Theorem 1. If a positive sequence $\omega(m) \uparrow$ is such that

$$\sum_{m=1}^{\infty} \frac{1}{m\omega(m)} = \infty,$$

then the sequence $\omega(m)$ cannot be a Weyl multiplier for unconditional convergence almost everywhere of series with respect to the Haar system $\{\chi_m(x)\}$.

From this theorem and from Theorem B it follows:

Theorem 2. The Haar system $\{\chi_m(x)\}$ has no exact Weyl multiplier for unconditional convergence almost everywhere.

Theorem 3. Not every orthonormal system has an exact Weyl multiplier for unconditional convergence.

Thus we see that if the question of the existence of an exact Weyl convergence multiplier for every orthonormal system is still open, then the same question for unconditional convergence has a negative solution.

The assertion constituting Theorem 2 leads to the question whether it will be true for any complete orthonormal system of functions. At present there is no answer to this question.

Further, since the Haar system has no exact Weyl multiplier for unconditional convergence, it is natural to try to find other criteria of unconditional convergence. In this direction we have obtained a certain final result. Namely, the following is true (see (7), Theorem 5.7′):

Theorem C. If $f(x) \in L^2(0,1)$ and a_m are its Fourier coefficients with respect to the Haar system, then the condition

$$\sum_{m=1}^{\infty} \frac{|a_m|}{\sqrt{m}} < \infty$$

implies unconditional convergence almost everywhere on $[0,1]$ of the series

$$\sum_{m=1}^{\infty} a_m \chi_m(x), \tag{1}$$

whereas the condition

$$\sum_{m=1}^{\infty} |a_m| \, \tau(m) < \infty$$

for a fixed positive sequence $\tau(m) = o(1/\sqrt{m})$ does not imply unconditional convergence almost everywhere of the series (1) for some $f(x) \in L^2(0,1)$.

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[6] P. L. Ul' yanov, DAN, 138, No. 3, 556 (1961).
[7] P. L. Ul' yanov, UMN, 16, issue 3, 61 (1961).

Note: Figure translations are in progress. See original paper for figures.