ON ISOMORPHIC REPRESENTATIONS OF RINGS AND GROUPS

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Abstract

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As is known (see, for example, (^{1})), there exist groups whose regular representation decomposes in different ways into irreducible representations*; such is, for example, the discrete group of all transformations $y = ax + \beta$, where a ranges over all positive rational numbers and β over all rational numbers. This group is also remarkable in that its left regular representation is a factor representation (of type II_1) decomposing into mutually inequivalent irreducible representations. These circumstances show that the notions of irreducible representation and of equivalence of representations are not sufficiently adequate in questions of the general theory of representations. Below we propose the notions of isomorphism of representations and of a simple representation, which may prove useful in a number of questions of representation theory.

Let R be a complete, completely regular** ring; by a representation of the ring R we shall mean a symmetric homomorphism $x \to U_x$ of the ring R into the ring $B(\mathfrak{H})$ of all bounded linear operators in some Hilbert space $\mathfrak{H}$. As is known (see (^{2}), corollary of item 3, § 24), the image U_R of the ring R under such a homomorphism $x \to U_x$ is also a complete, completely regular ring. Two representations $x \to U_x$ and $x \to V_x$ of the ring R will be called **isomorphic** if $U_x = 0$ if and only if $V_x = 0$. In this case, putting in correspondence with each operator U_x the operator V_x, we obtain a symmetric isomorphism $U_x \to V_x$ of the rings U_R and V_R; consequently (see (^{2}), theorem 3, item 1, § 24),

$$|U_x| = |V_x| \tag{1}$$

for all $x \in R$; obviously, the converse assertion is also true. Thus:

I. *Representations $x \to U_x$, $x \to V_x$ of the ring R are isomorphic if and only if (1) holds for all $x \in R$.*

Remark 1. Obviously, for the isomorphism of representations $x \to U_x$, $x \to V_x$ it is sufficient that (1) hold on some set that is complete in R, for then it holds on all of R.

II. *Unitarily equivalent representations are isomorphic.*

III. *Quasi-equivalent*** representations are isomorphic.*

Proof. Let the representations $x \to U_x$ and $x \to V_x$ be quasi-equivalent, and let the representation $x \to W_x$ be the direct sum of a countable number of copies of the representation $x \to V_x$; then the representation $x \to U_x$ is equivalent to some part of the representation $x \to W_x$, and therefore $|U_x| \leq |W_x| = |V_x|$. Interchanging the roles of U_x and V_x, we conclude that also $|V_x| \leq |U_x|$, and therefore $|U_x| = |V_x|$.

In the case of irreducible representations, quasiequivalence coincides with equivalence, and both Propositions II and III give one and the same thing.

IV. *If one of the rings U_R, V_R consists of all completely continuous operators, while the other is irreducible, and if the representations $x \to U_x, x \to V_x$ are isomorphic, then they are unitarily equivalent.*

Proof. Suppose, for example, that U_R is the set $\mathfrak{C}_1$ of all completely continuous operators in the space $\mathfrak{H}_1$ of the representation $x \to U_x$, and let $\mathfrak{H}_2$ be the space of the irreducible representation $x \to V_x$. The isomorphism of U_R and V_R is then an irreducible representation of the ring $\mathfrak{C}$ in $\mathfrak{H}_2$, and is therefore unitarily equivalent to the identity representation $A \to A$ (see Theorem 2, § 22 in (^{2})). This means that the isomorphism of U_R onto V_R is generated by an isometric mapping of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$, i.e., the representations $x \to U_x,\ x \to V_x$ are unitarily equivalent.

Remark 2. The assertion of Proposition IV remains valid if U_R consists of all operators of the form $\lambda 1 + A,\ A \in \mathfrak{C}_1$. To see this, it is enough to restrict to $\mathfrak{C}_1$ the given isomorphism of U_R onto V_R. Thus, when $U_R = \mathfrak{C}_1$ or $U_R = \{\lambda 1 + A,\ A \in \mathfrak{C}_1\}$, isomorphism coincides with unitary equivalence. Below we shall see that in the general case this is no longer true even for irreducible representations.

A representation $x \to U_x$ of a ring R will be called **simple** if U_R is a simple ring. Let us note that an irreducible representation need not be simple; for example, the identity representation $A \to A$ of the ring $B(\mathfrak{H})$ is irreducible, but is not simple; on the other hand, the direct sum of two copies of one and the same finite-dimensional irreducible representation is an example of a simple, but not irreducible, representation. Obviously:

V. *A representation isomorphic to a simple representation is also a simple representation.*

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* This fact was first established by A. N. Kolmogorov (unpublished).

** We adhere here to the terminology given in (^{2}); in what follows R will always denote a complete, completely regular ring.

*** For the definition and basic properties of quasi-equivalent representations see (^{3}).

A ring R is called a ring of **simple type** if all its irreducible representations are simple. Every ring whose irreducible representations are given by completely continuous operators (a CCR-algebra in Kaplansky' s terminology (^{4})) is a ring of simple type, for then, for any irreducible representation $x \to U_x$, the ring U_R is the set of all completely continuous operators and hence is simple.

VI. *Every simple ring R is a ring of simple type; namely, all its representations are simple and isomorphic to one another.*

Proof. By the simplicity of R, every representation $x \to U_x$ of the ring R is an isomorphism, and therefore U_R is a simple ring. Further, if $x \to U_x,\ x \to V_x$ are representations of the ring R, then the correspondences $U_x \to x \to V_x$ generate an isomorphism of the rings U_R and V_R.

VII. *Every factor of finite class, and also every factor of class* III*, is a ring of simple type, and all its representations are isomorphic to one another.*

Indeed, every such factor is a simple ring (see (^{5}), § 4), and the assertion follows from VI.

It is now easy to give an example of isomorphic but nonequivalent irreducible representations. Let R be a factor of class II_1 (or III). By VII all its representations, in particular all its irreducible representations, are isomorphic to one another. On the other hand, all its irreducible representations cannot be equivalent to one another, for then R would be isomorphic to the ring of all completely continuous operators (see (^{2}), item 2, § 22 and (^{6})), which is impossible.

VIII. *If $x \to U_x,\ x \to V_x$ are simple representations of the ring R, not isomorphic to one another, then for any $A \in U_R,\ B \in V_R$ there exists an element $x \in R$ such that $U_x = A,\ V_x = B$.*

Proof. Put $I = \{x:\ x \in R,\ U_x = 0\}$. Obviously, I is a closed two-sided ideal in R. Let I_1 be the image of I under

mapping $x \to V_x$; since this mapping is a symmetric homomorphism of the full completely regular ring I, it follows that I_1 is also a complete and therefore closed (proper or improper) two-sided ideal in V_R. But, by assumption, V_R is a simple ring, so that either $I_1 = (0)$ or $I_1 = V_R$. The first case means that whenever $U_x = 0$, also $V_x = 0$; consequently, contrary to the assumption, the representations $x \to U_x,\ x \to V_x$ are isomorphic. Therefore $I_1 = R$, i.e., for every $B \in V_R$ there exists an element $x_1 \in R$ such that $U_{x_1} = 0,\ V_{x_1} = B$. Similarly, there exists an element $x_2 \in R$ such that $U_{x_2} = A,\ V_{x_2} = 0$. Putting $x = x_1 + x_2$, we obtain: $U_x = A,\ V_x = B$.

Let us apply Proposition VIII to continuous sums of representations. Let T be a bicompact space, μ a Borel measure on T, and let T be the support of the measure μ. Suppose that to each $t \in T$ there is assigned a representation $x \to U_x(t)$ in a Hilbert space $\mathfrak{H}(t)$, so that the following conditions are fulfilled:

1) $\mathfrak{H}(t)$ is a μ-measurable family;

2) $U_x(t)$ is a μ-measurable function of t for each $x \in R$;

3) $|U_x(t)|$ is a continuous function of t for each $x \in R$.

In what follows we shall, without any further qualifications, assume these conditions fulfilled, and denote by $x \to U_x$ the representation in the space

$$\mathfrak{H} = \int_T \mathfrak{H}(t) \sqrt{d\mu(t)},$$

which is the continuous sum of the representations

$$x \to U_x(t).$$

Theorem 1. *Let R be a ring with identity, and let $x \to U_x(t)$ be pairwise nonisomorphic simple representations of the ring R. Then U_R contains all functions $a(t)1$, where $a(t) \in C(t)$.*

Proof. By virtue of VIII, U_R satisfies all the hypotheses of Theorem 4, item 4, §26 in (?).

Corollary 1. *Suppose that all the hypotheses of Theorem 1 are fulfilled; then $(U_R)'$ consists of those and only those $A \in B(\mathfrak{H})$ that are representable in the form $A = \{A(t)\}$, where $A(t) \in (U(t)_R)'$ for almost every $t \in T$.*

The assertion follows directly from IV, item 5, §26 in (?) and Theorem 1.

Corollary 2. *Suppose that all the hypotheses of Theorem 1 are fulfilled and, moreover, all representations $x \to U_x(t)$ are irreducible; then $(U_R)'$ coincides with $(U_R)' \cap (U_R)''$ and is the set of all $A = \{a(t)1\}$, $a(t) \in L^\infty_\mu(T)$.*

Proof. By the irreducibility of the representation $x \to U_x(t)$, the ring $(U_R)'$ consists of scalars; therefore the second assertion follows directly from Corollary 1. But then $(U_R)'$ is commutative; consequently, $(U_R)' = (U_R)' \cap (U_R)''$.

Corollary 3. *Suppose that the same hypotheses as in Corollary 2 are fulfilled. Then $(U_R)''$ is a factor if and only if T consists of a single point.*

Indeed, only in this case is $L^\infty_\mu(T)$ the ring of scalars.

An essential strengthening of Corollary 3 is:

Theorem 2. *Let R be a ring with identity and let $x \to U_x(t)$ be a simple representation of the ring R for each $t \in T$. If then $(U_R)''$ is a factor, all representations $t \to U_x(t)$ are isomorphic to one another.*

Proof. Suppose the contrary; then there exist $t_1, t_2 \in T$ $(t_1 \neq t_2)$ such that the representations $x \to U_x(t_1)$, $x \to U_x(t_2)$ are nonisomorphic. By virtue of I this means that there exists $x_0 \in R$ such that $|U_{x_0}(t_1)| \neq |U_{x_0}(t_2)|$. But then, by condition 3), there exist neighborhoods $\mathfrak{A}(t_1)$, $\mathfrak{A}(t_2)$ of the points t_1, t_2 such that $|U_{x_0}(t')| \neq |U_{x_0}(t'')|$ for $t' \in \mathfrak{A}(t_1)$, $t'' \in \mathfrak{A}(t_2)$.

Shrinking these neighborhoods if necessary, we may assume that

$$|U_{x_0}(t')| \neq |U_{x_0}(t'')| \quad \text{for } t' \in \overline{\mathfrak{A}(t_1)}, \quad t'' \in \overline{\mathfrak{A}(t_2)}, \tag{2}$$

where the bar denotes closure. Put $T_1 = \overline{\mathfrak{A}(t_1)} \cup \overline{\mathfrak{A}(t_2)}$, and denote by $\mathfrak{H}_1$ the set of all $\xi = \{\xi(t)\} \in \mathfrak{H}$ such that $\xi(t) = 0$ for

$t \in \bar{T}_1$. Then $\mathfrak{H}_1$ is an invariant subspace for U_R. Let U_{R1} be the restriction of U_R to $\mathfrak{H}_1$; then $(U_{R1})''$ is also a factor.

By virtue of (2), the representations $x \to U_x(t')$, $x \to U_x(t'')$ are nonisomorphic for $t' \in \mathfrak{A}(t_1)$, $t'' \in \mathfrak{A}(t_2)$. Therefore, using Proposition VIII and arguing as at the beginning of the proof of Theorem 4, item 4, § 26 in ([2]), we conclude that in U_{R1} there exists a function $A = \{A(t)\}$ such that $A(t) = 0$ for $t \in \mathfrak{A}(t_1)$, and $A(t) = 1$ for $t \in \mathfrak{A}(t_2)$.

This function belongs to the center of the ring U_{R1}, and hence also of the ring $(U_{R1})''$. But then the center of the ring $(U_{R1})''$ does not consist solely of scalars, i.e., contrary to what was said above, $(U_{R1})''$ is not a factor.

Remark 3. Theorems 1, 2 and Corollaries 1-3 are easily extended (with the appropriate changes in their formulations) to locally bicompact spaces T (see, in this connection, Remark 3 on p. 308 in ([2])); we have restricted ourselves to the case of bicompact T in order to simplify the exposition.

Remark 4. The notions introduced here of isomorphy of representations and of a simple representation are easily carried over to unitary representations of a locally bicompact group G. Let $R(G)$ denote the completion of the group ring $L^1(G)$ in the minimal regular norm. Representations $g \to U_g$, $g \to V_g$ are called isomorphic if the corresponding representations $x \to U_x$, $x \to V_x$ of the ring $R(G)$ are isomorphic; the representation $g \to U_g$ is called simple if the corresponding representation $x \to U_x$ of the ring $R(G)$ is simple. Finally, the group G is called a group of simple type if $R(G)$ is a ring of simple type.

It remains now to apply the preceding results to the ring $R(G)$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.