On a Class of Completely Continuous Operators

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Abstract

Full Text

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In the present note we adhere to the terminology and notation of ([1]). In particular, by $\mathfrak{S}_\infty$ is meant the Banach space of all completely continuous operators acting in a separable Hilbert space $\mathfrak{H}$ with the usual norm $|A|_\infty = \sup_{f\in\mathfrak{H}}(|Af|/|f|)$; by $\mathfrak{S}_p$ $(1 \le p < \infty)$, the Banach space of all operators $A \in \mathfrak{S}_\infty$ for which $(|A|_p)^p = \sum s_n^p(A) = \mathrm{Sp}((A^*A)^{p/2}) < \infty$, where $s_n(A)$ is the sequence, numbered in decreasing order with multiplicities taken into account, of the eigenvalues of the operator $(A^*A)^{1/2}$; and by $\mathfrak{S}_\omega$, the Banach space of operators $A \in \mathfrak{S}_\infty$ for which $|A|_\omega = \sum(2n-1)^{-1}s_n(A) < \infty$. It is obvious that $\mathfrak{S}_p \subset \mathfrak{S}_\omega$ for any p $(1 \le p < \infty)$.

The set $\hat{\mathfrak{S}}_\omega$ of all self-adjoint operators in $\mathfrak{S}_\omega$ is a real subspace of the space $\mathfrak{S}_\omega$. In the space $\hat{\mathfrak{S}}_\omega$ we introduce a new norm $|H|_{\omega,K^*}$, topologically equivalent to the original one, by setting $|H|_{\omega,K^*} = |H^+|_\omega + |H^-|_\omega$, where H^+ and H^- are mutually orthogonal nonnegative operators whose difference is equal to H.

M. S. Brodskii ([2]) showed that every Volterra operator A (i.e. every completely continuous operator A with the single spectral point $\lambda = 0$) admits a representation convergent in the norm of $\mathfrak{S}_\infty$ *

$$A = 2i\int_{\mathfrak{P}} PX\,dP \tag{1}$$

through its imaginary Hermitian component X and a maximal eigenchain $\mathfrak{P}$. Conversely, if for some $X = X^* \in \mathfrak{S}_\infty$ the integral (1) converges in the norm of $\mathfrak{S}_\infty$, then A is the unique Volterra operator possessing the eigenchain $\mathfrak{P}$ and imaginary component X. We shall denote the real component of the integral (1) by $\mathfrak{S}(\mathfrak{P}, X)$. A necessary condition for convergence of the integral (1) is the condition

$$(P-Q)X(P-Q) = 0, \tag{2}$$

where (P, Q) is an arbitrary break of the chain $\mathfrak{P}$. The sufficiency of this condition for $X \in \mathfrak{S}_1$ was proved by M. S. Brodskii ([2]), and for $X \in \mathfrak{S}_p$ $(p > 1)$ by I. Ts. Gohberg and M. G. Krein (see ([1],[3])).

Theorem 1. *For every operator $X \in \mathfrak{S}_\omega$ and every chain $\mathfrak{P}$ satisfying condition (2), the integral (1) converges. Moreover, the relation holds*

$$\sup|G|_\infty = \frac{2}{\pi}\sum_{j=1}^{\infty}\frac{s_j(H^+)+s_j(H^-)}{2j-1} = \frac{2}{\pi}|H|_{\omega,K^*}, \tag{3}$$

* Here we use the definition and notation for the integral of triangular truncation proposed by I. Ts. Gohberg and M. G. Krein ([1]).

where the supremum on the left is taken over all Volterra operators $A = G + iH$ with fixed imaginary component $H = H^* \in \mathfrak{S}_\omega$.

On the basis of the Fan–Tzye inequalities (Ky Fan ([4])) one first establishes

Lemma 1. Let $H = H^* \in \mathfrak{S}_\omega$; then

$$\sup\sum_{n=-\infty}^{\infty}(2n-1)^{-1}(H\varphi_n,\varphi_n) = |H|_{\omega,K^*}, \tag{4}$$

where the least upper bound is taken over all possible orthonormal systems $\{\varphi_j\}_{-\infty}^{\infty}$ from $\mathfrak{H}$.

Proceeding to the proof of Theorem 1, we note that it is enough to prove its first assertion for self-adjoint operators.

Without loss of generality, we shall suppose that the chain $\mathfrak{P}$ is continuous. First assume that $H = H^*$ is a finite-dimensional operator. For $X = H$, the integral (1) converges to some Volterra operator $A = G + iH$, which belongs to $\mathfrak{S}_2$. Using the integral (1), construct a Volterra operator $B = K + iL$ with its own chain $\mathfrak{P}$ and real component $K = (\cdot,\varphi)\varphi$, where φ is a given unit vector from $\mathfrak{H}$. The completely continuous operator AB has the continuous proper chain $\mathfrak{P}$ and, consequently, is also Volterra. Since $A, B \in \mathfrak{S}_2$, it follows that $AB \in \mathfrak{S}_1$, and by a theorem of V. B. Lidskii ([4]), $\mathrm{Sp}(AB) = 0$. Taking the real components of both sides of this equality, we obtain $\mathrm{Sp}(GK - HL) = 0$, or

$$\mathrm{Sp}(GK) = \mathrm{Sp}(HL). \tag{5}$$

By a theorem of M. S. Livshits (see, for example, ([3])) the operator B is unitarily equivalent to an inessential extension of the integration operator I in $\mathcal{L}^2(0,1)$:

$$If = 2\int_0^x f(t)\,dt.$$

It follows that the nonzero eigenvalues of the operator L are equal to $2/\pi(2j-1)$ $(j = 0, \pm1, \ldots)$.

Now (5) can be rewritten in the form

$$(G\varphi, \varphi) = \frac{2}{\pi} \sum_{j=-\infty}^{\infty} (2j-1)^{-1} (H\varphi_j, \varphi_j), \tag{6}$$

where $\{\varphi_j\}$ are the eigenvectors of the operator L. Applying Lemma 1 and taking on the left in (5) the least upper bound with respect to φ, we obtain

$$|G|_\infty \leq \frac{2}{\pi} |H|_{\omega, K^*}. \tag{7}$$

Let now $H \in \mathfrak{S}_\omega$. Take a sequence of finite-dimensional operators $H_n = H_n^*$ such that $|H_n - H|_\omega \to 0$. By virtue of (7) we have

$$|\mathfrak{S}(\mathfrak{P}, H_n) - \mathfrak{S}(\mathfrak{P}, H_m)| \to 0;$$

therefore the operators $\mathfrak{S}(H_n) + iH_n$ converge to some Volterra operator $A = G + iH$, which has the proper chain $\mathfrak{P}$ and, by the above-mentioned theorem of M. S. Brodskii, admits a representation in the form of an integral (1) convergent in the norm of $\mathfrak{S}_\infty$. Passing in the inequality

$$|\mathfrak{S}(H_n)| \leq \frac{2}{\pi} |H_n|_{\omega, K^*}$$

to the limit as $n \to \infty$, we obtain inequality (7). Since A is the unique operator with imaginary component H and proper chain $\mathfrak{P}$, this inequality may be regarded as established for any Volterra operator $A = G + iH$ with $H \in \mathfrak{S}_\omega$. In exactly the same way one can prove for A the validity of relation (6).

It remains to verify that for every $\varepsilon > 0$ there exists a Volterra operator $A = G + iH$ with given imaginary component H ($\in \mathfrak{S}_\omega$), such that

$G \, |> \frac{2}{\pi} |H|_{\omega, K^*} - \varepsilon$. By Lemma 1 there exists an orthonormal basis $\{\varphi_j\}_{-\infty}^{\infty}$ for which (4) is fulfilled. Take an operator $B = K + iL$, unitarily equivalent to the integration operator and such that $L\varphi_n = (2/\pi(2n-1))\varphi_n$ $(n = 0, \pm 1, \ldots)$. Let $\mathfrak{P}$ be its proper chain. Putting $G = \mathfrak{S}(\mathfrak{P}, H)$ and using (6), we obtain

$$|G| \geq (G\varphi, \varphi) = \frac{2}{\pi} \sum_{-\infty}^{\infty} (2j-1)^{-1} (H\varphi_j, \varphi_j) > \frac{2}{\pi} (|H|_{\omega, K^*} - \varepsilon).$$

Theorem 1 is completely proved.

Equality (3) led the author to suppose that the belonging of the operator $X \in \mathfrak{S}_\infty$ to the class $\mathfrak{S}_\omega$ is not only a sufficient but also a necessary condition for the integral (1) to converge for any continuous chain $\mathfrak{P}$. The validity of this supposition was proved by I. Ts. Gokhberg and M. G. Krein, who also communicated to the author the following proposition.

Let $\{\varphi_j\}_{-\infty}^{\infty}$ be some orthonormal system. Then there exists a continuous chain $\mathfrak{P}$ such that, for any completely continuous self-adjoint operator

$$X = \sum_j \lambda_j(\,\cdot\,,\varphi_j)\varphi_j \quad \left((2j-1)\lambda_j > 0;\; j = 0, \pm 1, \pm 2, \dots;\; \sum_{j=-\infty}^{\infty} \frac{\lambda_j}{2j-1} = \infty\right)$$

the integral (1) diverges (even in the sense of weak convergence). Conversely, for any continuous chain $\mathfrak{P}$ one can construct an orthonormal system $\{\varphi_j\}_{-\infty}^{\infty}$ such that the integral (1) diverges for any operator X only of the indicated type.

The possibility of divergence of the integral (1) for some $X \in \mathfrak{S}_\infty$, under a special choice of the chain $\mathfrak{P}$, was discovered by M. S. Brodskii ([2]).

It can be proved that for every operator $X = X^* \in \mathfrak{S}_\infty$ there is always a continuous chain $\mathfrak{P}$ for which the integral (1) converges.

2. L. A. Sakhnovich ([6]) showed that if, for a bounded operator with real spectrum $A = G + iH$, the imaginary component $H \in \mathfrak{S}_2$, then this operator has a sufficiently rich supply of invariant subspaces. I. Ts. Gokhberg and M. G. Krein, and independently of them the author, showed that this fact holds for $H \in \mathfrak{S}_p$.

There is the following more general and complete proposition:

Theorem 2. *An operator $A = G + iH$, $H \in \mathfrak{S}_\omega$, having real spectrum, is an S-operator in the sense of* ([7]).

Let us explain that an operator A with real spectrum is called an S-operator if it possesses the following properties. For every finite interval Δ of the real axis there exists a subspace $L(\Delta)$, invariant with respect to the operator A, such that: a) on $L(\Delta)$ the operator A is everywhere defined and bounded; b) the spectrum of the part of the operator A induced on $L(\Delta)$ consists of the intersection of the spectrum of the operator A with the interval Δ and, possibly, the endpoints of the interval Δ; c) every invariant subspace on which the operator A is everywhere defined, bounded, and has as spectrum a part of the segment Δ, is contained in $L(\Delta)$; d) the system of invariant subspaces corresponding to any covering of the real axis by intervals is complete in $\mathfrak{H}$.

Apparently, Theorem 2 admits the following converse: for every operator $H = H^* \in \mathfrak{S}_\omega$ there is an operator $G = G^*$ such that the operator $A = G + iH$, on each of its invariant subspaces, has a spectrum coinciding with the spectrum of the whole operator A, which in this case is real and consists of more than one point.

Theorem 2 is proved on the basis of Theorem 3 from ([7]) and the following lemma.

Lemma 2. The resolvent of an operator with real spectrum $A = G + iH$, where $G = G^*$ (in general, unbounded), and $H = H^* \in \mathfrak{S}_\infty$, satisfies the following estimate:

$$\ln |R_\lambda| = \ln |(A - \lambda I)^{-1}| \leq C\,[1 + |\operatorname{Im}\lambda|^{-2} n(2|\operatorname{Im}\lambda|^{-1})],$$

where $n(t)$ denotes the number of $1/s_k(H)$ not exceeding t.

3. Using other estimates of the resolvent, one can prove the following theorem:

Theorem 3. The system of eigenvectors and associated vectors of the operator $A = H(I + T)$, where $H = H^* \in \mathfrak{S}_\infty$, and $T \in \mathfrak{S}_\omega$, is complete in its range.

This theorem can easily be given a form similar to that of the theorem of M. V. Keldysh from [8], namely:

If the equation $y = Ay + \lambda Hy$ has a discrete spectrum, $H = H^* \in \mathfrak{S}_\infty$ is a complete operator (see [8]), $A \in \mathfrak{S}_\omega$, then the system of eigenvectors and associated vectors of this equation is complete in $\mathfrak{H}$.

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REFERENCES

1. I. Ts. Gokhberg, M. G. Krein, DAN, **137**, No. 5 (1961).

2. M. S. Brodskii, Uspekhi Mat. Nauk, **16**, issue 1 (1961).

3. Fan Ky, Proc. Nat. Acad. Sci., **1**, 35 (1949); **2**, 36 (1950).

4. V. B. Lidskii, DAN, **125**, No. 3 (1958).

5. I. Ts. Gokhberg, M. G. Krein, DAN, **128**, No. 2 (1959).

6. L. A. Sakhnovich, Izv. Vyssh. Uchebn. Zaved., Mathematics, No. 4, (11) (1959).

7. Yu. I. Lyubich, V. I. Matsaev, DAN, **13**, No. 1 (1960).

8. M. V. Keldysh, DAN, **77**, No. 1 (1951).

Note: Figure translations are in progress. See original paper for figures.