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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

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### ON THE COMPLETENESS OF CLOSE SYSTEMS IN A COUNTABLY NORMED SPACE

*(Presented by Academician V. I. Smirnov on 6 XII 1960)*

M. A. Evgrafov proved several theorems on the completeness of close systems in the analytic space  $\mathfrak{A}(D)$  under the assumption that the domain  $D$  is simply connected and that the original system is continuous in the closure  $\bar{D}$  of the domain  $D$  (<sup>(1)</sup>, Ch. II, § 1, Theorems 1, 2, 3). In the present note we shall prove analogous theorems for complete countably normed spaces, from which, in particular, a generalization of the cited theorems will follow for the case of an arbitrary analytic space on an open set without the assumption that the original system is bounded.

Let  $X$  be a complete countably normed space,  $X = \lim_{\text{pr}} B_n$  (<sup>(2)</sup>);  $B_n$  ( $n = 0, 1, \dots$ ) is a decreasing sequence of Banach spaces with monotonically increasing norms  $\|x\|_n$  ( $x \in X$ ). Then (<sup>(2)</sup>)  $X = \bigcap_n B_n$ .

**Theorem 1.** If the system  $\{x_n\}_0^\infty$  is complete in  $X$  and  $\|x_n\|_n = s_n \neq 0$  ( $n = 0, 1, 2, \dots$ ), then the system

$$y_n = x_n + \sum_{k=0}^{\infty} \alpha_{kn} x_k \quad (n = 0, 1, \dots) \quad (1)$$

is complete in  $X$ , if

$$\sup_n \sum_{k=0}^{\infty} |\alpha_{kn}| s_k s_n^{-1} = \theta < 1. \quad (2)$$

**Proof.** Let  $B$  be the Banach space of all sequences  $b = \{b_n\}_0^\infty$  such that  $\|b\| = \sum_n |b_n| s_n < \infty$ . Denote the unit vectors of  $B$  by

$$e_n = \left\{ \frac{1}{s_n} \delta_{kn} \right\}_{k=0}^{\infty},$$

and let  $\eta_n = s_n e_n = \{\delta_{kn}\}_{k=0}^{\infty}$ . Consider the operator  $A$ , defined at first on  $\{\eta_n\}$ :  $A\eta_n = \{\alpha_{kn}\}_k$  ( $n = 0, 1, \dots$ ). Then

$$\|Ae_n\| = \sum_k |\alpha_{kn}| s_k s_n^{-1} \leq \theta < 1.$$

Extend  $A$  linearly and continuously to the whole space  $B$ ; then  $\|A\| = \theta < 1$ , and, consequently, the operator  $E + A$  is a linear homeomorphism of the space  $B$ . Introduce also an operator  $P$ , acting from  $B$  into  $X$  according to the formula

$$P\{b_n\} = \sum_n b_n x_n.$$

From

$$|b_n| \|x\|_m \leq |b_n| s_n \|x_n\|_m s_n^{-1} \leq |b_n| s_n \quad (n > m)$$

it follows that

$$\sum_n b_n x_n \in B_m$$

for every  $\{b_n\} \subset B$  and any  $m$ . Consequently,  $\sum_n b_n x_n \in X$ , and  $P$  maps  $B$  into  $X$ . The operator  $P$  is continuous. Indeed, let  $V$  be an arbitrary neighborhood of zero in  $X$ , consisting of all  $x \in X$  with  $\|x\|_{n_0} \leq 1$ , and let  $s'_k = \|x_k\|_{n_0}$  ( $k = 0, 1, \dots, n_0 - 1$ ). Choose  $\varepsilon > 0$  so that

$$\varepsilon \left( 1 + \sum_{k=0}^{n_0-1} s'_k s_k^{-1} \right) \leq 1.$$

Let now  $W$  be the neighborhood of zero in  $B$  consisting of all  $b = \{b_n\}$  with

$$\sum_n |b_n| s_n < \varepsilon.$$

Then for  $b \in W_\infty$  we have

$$\begin{aligned} \left\| \sum b_k x_k \right\|_{n_0} &\leq \sum_{k=0}^{n_0-1} |b_k| \|x_k\|_{n_0} + \sum_{k=n_0}^{\infty} |b_k| \|x_k\|_k \leq \\ &\leq \sum_{k=0}^{n_0-1} |b_k| s_k s'_k s_k^{-1} + \sum_{k=n_0}^{\infty} |b_k| s_k \leq \varepsilon \left( \sum_{k=0}^{n_0-1} s'_k s_k^{-1} + 1 \right) \leq 1, \end{aligned}$$

i.e.  $Ab \in V$ , if  $b \in W$ .

From  $P\eta_n = x_n$  ( $n = 0, 1, \dots$ ) there follows the density of the range  $R_P$  of the operator  $P$  in  $X$ ,  $\overline{R_P} = X$ . On the other hand,  $R_{P(E+A)} = R_P$ . If  $x \in R_P$ , then there exists  $b \in B$  such that  $x = P(E+A)b$ . Since  $b$  in  $B$  is the limit of linear combinations of the vectors  $\eta_n$  ( $n = 0, 1, \dots$ ), and  $P(E+A)\eta_n = y_n$  ( $n = 0, 1, \dots$ ), then, by the continuity of the operator  $P(E+A)$ ,  $x$  is the limit of linear combinations of  $\{y_n\}$ , whence it follows, in view of  $\overline{R_P} = X$ , that the system  $\{y_n\}$  is complete in  $X$ .

**Corollary.** If instead of condition (2) the condition

$$\sup_n \sum_{k=0}^{\infty} |\alpha_{kn}| s_k s_n^{-1} = \theta < \infty, \quad (2')$$

is satisfied, then the system

$$y_n = x_n + \lambda \sum_{k=0}^{\infty} \alpha_{kn} x_k \quad (n = 0, 1, \dots) \quad (1')$$

is complete in  $X$  for all  $\lambda$  with  $|\lambda| < 1/\theta$ .

**Theorem 2.** If the system  $\{x_n\}_0^{\infty}$  is complete in  $X$ , then the system

$$y_n = x_n + \sum_{k=n+1}^{\infty} \alpha_{kn} x_k \quad (n = 0, 1, \dots) \quad (3)$$

is complete in  $X$  provided that

$$\sum_{k=n+1}^{\infty} |\alpha_{kn}| s_k s_n^{-1} < \infty \quad (n = 0, 1, \dots), \quad \lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} |\alpha_{kn}| s_k s_n^{-1} < 1. \quad (4)$$

**Proof.** In the present case  $\alpha_{kn} = 0$  for  $k \leq n$ . Suppose that for  $n \geq N$

$$\sup_{n \geq N} \sum_{k=n+1}^{\infty} |\alpha_{kn}| s_k s_n^{-1} = \theta < 1.$$

Then the operator  $A$ , defined above, can be decomposed into the sum of two operators  $A_1 + A_2$ , of which  $A_1$ , with matrix  $[\alpha_{kn}]$  subject to the additional condition  $\alpha_{kn} = 0$  ( $n \geq N$ ), is a bounded finite-dimensional operator in  $B$ , and  $A_2$ , with matrix  $[\alpha_{kn}]$ ,  $\alpha_{kn} = 0$  ( $n < N$ ), has norm  $\|A_2\| < 1$ . The operator  $E + A = (E + A_2) + A_1$  is the sum of a finite-dimensional and an invertible operator in  $B$ , and therefore<sup>(3)</sup> Fredholm theory is applicable to it. Suppose now that for some  $b \in B$ ,

$$b = \sum_n b_n \eta_n, \quad (E + A)b = 0, \quad \text{i.e.}$$

$$\begin{aligned} (E + A)b &= \sum_n b_n (E + A)\eta_n = \sum_n b_n \left( \eta_n + \sum_{k=0}^{\infty} \alpha_{kn} \eta_k \right) = \\ &= b_0 \eta_0 + (b_1 + b_0 \alpha_{10}) \eta_1 + (b_2 + \alpha_{20} b_0 + \alpha_{21} b_1) \eta_2 + \dots = 0. \end{aligned}$$

Hence we obtain  $b = 0$ . On the basis of Fredholm theory we conclude that  $E + A$  is a linear homeomorphism of the space  $B$ . The argument is completed as in the proof of Theorem 1.

**Corollary.** If the sequence  $\{x_n\}_0^\infty$  is complete in  $X$ ,

$$\lim_{n \rightarrow \infty} \sum_{k=n+1}^{\infty} |a_{kn}| s_k s_n^{-1} = N < \infty, \quad \sum_{k=n+1}^{\infty} |a_{kn}| s_k s_n^{-1} < \infty \quad (n = 0, 1, \dots),$$

then the system

$$y_n = x_n + \lambda \sum_{k=n+1}^{\infty} a_{kn} x_k$$

is complete in  $X$  for  $|\lambda| < 1/N$ .

**Theorem 3.** If the system  $\{x_n\}_0^\infty$  is complete in  $X$ , then the system

$$y_n = x_n + \lambda \sum_{k=0}^{\infty} a_{kn} x_k \quad (n = 0, 1, \dots) \quad (5)$$

is complete in  $X$  for all  $\lambda$ , with the exception of a discrete set of values of  $\lambda$ , provided one of the following conditions is fulfilled:

$$\begin{aligned} 1) \quad & \sum_{k=0}^{\infty} |a_{kn}| s_k s_n^{-1} < \infty \quad (n = 0, 1, \dots), \quad \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} |a_{kn}| s_k s_n^{-1} = 0; \\ 2) \quad & \sup_n \sum_{k=0}^{\infty} |a_{kn}| s_k s_n^{-1} < \infty, \quad \lim_{n \rightarrow \infty} \sup_n \sum_{k=N}^{\infty} |a_{kn}| s_k s_n^{-1} = 0. \end{aligned} \quad (6)$$

Under these assumptions the operator  $A$  is completely continuous in  $B$ , and therefore  $E + \lambda A$  is a linear homeomorphism of  $B$ , except for a discrete set of values of  $\lambda$ .

As an application, consider an arbitrary nonempty open set  $G$  of the Riemann sphere  $\Omega$ ,  $G \neq \Omega$ , and the corresponding analytic space  $X = \mathfrak{U}(G)$  <sup>(4)</sup>. Let  $G_n$  ( $n = 0, 1, \dots$ ) be an increasing sequence of open sets, each of which is contained in a finite number of connected components of  $G$ ,  $\overline{G}_n \subset G_{n+1}$ ,  $\bigcup_n G_n = G$ . Introduce Banach spaces  $B_n$  ( $n = 0, 1, \dots$ ) of all locally analytic functions  $x(z)$  in  $G_n$ , continuous on  $\overline{G}_n$ , with

$$\|x(z)\|_n = \sup_{z \in \overline{G}_n} |f(z)|.$$

Then, according to what was indicated above, we have:

**Theorem 1'.** If the system  $\{x_n(z)\}_0^\infty$ ,  $x_n(z) \in \mathfrak{U}(G)$  ( $n = 0, 1, \dots$ ), is complete in  $\mathfrak{U}(G)$  and

$$\sup_{z \in \overline{G}_n} |x_n(z)| = s_n \neq 0,$$

then the system of functions

$$y_n(z) = x_n(z) + \sum_{k=0}^{\infty} a_{kn} x_k(z) \quad (n = 0, 1, \dots) \quad (7)$$

is complete in  $\mathfrak{U}(G)$  when condition (2) is fulfilled.

**Theorem 2'.** Let the system  $\{x_n(z)\}_0^\infty$  be complete in  $\mathfrak{U}(G)$ . Then the system

$$y_n(z) = x_n(z) + \sum_{k=n+1}^{\infty} a_{kn} x_k(z) \quad (n = 0, 1, \dots) \quad (8)$$

is complete in  $\mathfrak{U}(G)$  when the conditions (4) are fulfilled.

**Theorem 3'.** If the system  $\{x_n(z)\}_0^\infty$  is complete in  $\mathfrak{U}(G)$ , then the system (7) is complete in  $\mathfrak{U}(G)$  for all  $\lambda$ , with the exception of a discrete set of values of  $\lambda$ , if the matrix  $[a_{kn}]$  satisfies one of the conditions (6).

The proved criteria of completeness can also be applied in analytic spaces of functions of many complex variables on polycylindrical domains and in some spaces of infinitely differentiable functions.

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*Note: Figure translations are in progress. See original paper for figures.*

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