APPROXIMATE SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR A NONLINEAR ORDINARY DIFFERENTIAL EQUATION OF SECOND ORDER WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE ON THE BASIS OF NEWTON' S METHOD

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Abstract

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APPROXIMATE SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR A NONLINEAR ORDINARY DIFFERENTIAL EQUATION OF SECOND ORDER WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE ON THE BASIS OF NEWTON' S METHOD

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The problem of determining the temperature field of a cooled blade of an aviation gas turbine reduces to the following boundary-value problem ([1]) on the interval $[0, 1]$:

$$\varepsilon^2 \frac{d}{dt}\left[f(t)\frac{dx}{dt}\right] - \Psi(x,t) = 0; \qquad x(0) = \alpha, \quad x(1) = \beta. \tag{1}$$

Assume the function $\Psi(u,t)$ to be continuous and to have a continuous second derivative with respect to u in the domain $0 \le t \le 1,\ |u - x_0(t)| \le \delta$. The function $f(t)$ is twice continuously differentiable and strictly positive on $[0, 1]$. As the initial approximation we take the twice continuously differentiable function $x_0(t)$

$$x_0(t) = \begin{cases} x^*(t) & \text{on } [\varepsilon,\, 1-\varepsilon], \\ \bar{x}_0(t) & \text{on } [0,\varepsilon],\ [1-\varepsilon, 1]. \end{cases}$$

Here $x^*(t)$ is the solution of equation (1) for $\varepsilon = 0$, and $\bar{x}_0(t)$ is a twice continuously differentiable function satisfying the boundary conditions of problem (1).

To apply the general theory of Newton' s method, we shall regard the differential equation (1) as a functional equation in the space $x = C^2$ of twice continuously differentiable functions satisfying the boundary conditions, with norm

$$\|\dot{x}\| = \lambda \max_{t\in[0,1]} \left|\varepsilon^2 \frac{d}{dt}\left[f(t)\frac{dx}{dt}\right]\right| + \max_{t\in[0,1]} |x(t)|, \tag{2}$$

where $\lambda > 0$ will be determined below.

As the space y we consider the space C of functions continuous on the interval $[0,1]$, with the usual definition of the norm:

$$\|y\| = \max_{t\in[0,1]} |y(t)|.$$

Consider the operation $\mathcal{P}$,

$$y = \mathcal{P}(x), \qquad y(t) = \varepsilon^2 \frac{d}{dt}\left[f(t)\frac{dx}{dt}\right] - \Psi(x(t),t). \tag{3}$$

It is not difficult to verify that $\mathcal{P}$ maps the sphere Ω_0, $\|x - x_0\| \le \delta$, into the space C and has in this sphere continuous first and sec-

of second order. In this case

$$\mathfrak{P}'(z)(\Delta x)(t) = \varepsilon^2 \frac{d}{dt}\left[f(t)\frac{d(\Delta x)}{dt}\right] - \Psi'_u(z(t),t)\Delta x(t); \tag{4}$$

$$\mathfrak{P}''(z)(\Delta x, \widetilde{\Delta x})(t) = -\Psi''_{u^2}(z(t),t)\Delta x(t)\cdot\widetilde{\Delta x}(t). \tag{5}$$

Consequently, from (4) the element $\Delta x = \Gamma_0(y)$ is the solution of the boundary-value problem

$$\varepsilon^2 \frac{d}{dt}\left[f(t)\frac{d(\Delta x)}{dt}\right] - \Psi'_u(x_0(t),t)\Delta x = y(t); \tag{6}$$

$$\Delta x(0) = 0, \qquad \Delta x(1) = 0.$$

Consider the homogeneous equation corresponding to (6), and by the substitution
$\Delta x = u(t)/\sqrt{f(t)}$ reduce it to the form

$$\varepsilon^2 u''(t) - \left[q^2(t) + \varepsilon^2 r(t)\right] u(t) = 0. \tag{7}$$

Here

$$q^2(t) = -\frac{\Psi'_u(x_0(t),t)}{f(t)}, \qquad r(t) = \frac{f''(t)}{2f(t)} - \frac{f'^2(t)}{4f^2(t)}. \tag{8}$$

Under the assumption of uniform boundedness of the exact solution of equation (7), it is easy to show that the approximate solution of equation (7), with accuracy up to a quantity $O(\varepsilon^3)$, has the form

$$u(t) = \frac{c_1}{\sqrt{q(t)}} \exp\left(\frac{1}{\varepsilon}\int_0^t q(\tau)\,d\tau\right) + \frac{c_2}{\sqrt{q(t)}} \exp\left(-\frac{1}{\varepsilon}\int_0^t q(\tau)\,d\tau\right). \quad (9)$$

Now the linear operation Γ, close to $\Gamma_0 = [\mathfrak{P}'(x_0)]^{-1}$, is easily determined. The Green' s function for the operator

$$\varepsilon^2 \frac{d}{dt}\left[f(t)\frac{d(\Delta x)}{dt}\right] - \Psi'_u(x_0(t),t)\Delta x$$

under the conditions $\Delta x(0) = \Delta x(1) = 0$, taking (8) and (9) into account, has the form:

$$G(t,s) = -\frac{\varepsilon f(0)}{2}\left[\exp\left(\frac{1}{\varepsilon}\int_0^t \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right) - \exp\left(-\frac{1}{\varepsilon}\int_0^t \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right)\right] \times$$
$$\times \left[\exp\left(-\frac{1}{\varepsilon}\int_0^s \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right) - \exp\left(-\frac{2}{\varepsilon}\int_0^1 \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right)\exp\left(\frac{1}{\varepsilon}\int_0^s \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right)\right] \times$$
$$\times \left\{[\Psi'_u(x_0,t)f(t)]^{1/4}[\Psi'_u(x_0,s)f(s)]^{1/4}\left[1 - \exp\left(-\frac{2}{\varepsilon}\int_0^1 \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right)\right]\right\}^{-1}$$

for $0 \le t \le s$;

(10)

$$G(t,s) = -\frac{\varepsilon f(0)}{2}\left[\exp\left(-\frac{1}{\varepsilon}\int_0^t \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right) - \exp\left(-\frac{2}{\varepsilon}\int_0^1 \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right) \times\right.$$
$$\left.\times \exp\left(\frac{1}{\varepsilon}\int_0^t \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right)\right]\left[\exp\left(\frac{1}{\varepsilon}\int_0^s \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right) - \exp\left(-\frac{1}{\varepsilon}\int_0\right.\right.$$
$$\times \left\{[\Psi'_u(x_0,t)f(t)]^{1/4}[\Psi'_u(x_0,s)f(s)]^{1/4}\left[1 - \exp\left(-\frac{2}{\varepsilon}\int_0^1 \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right)\right]\right\}^{-1}$$

for $s \le t \le 1$.

Let us coordinate the norm of X with the norm of Y, so that the operation G carries out an isometry between these spaces. Since the boundary-value problem (6) has a unique solution, we have

$$\Delta x = \int_0^1 G(t,s) y(s)\, ds. \tag{11}$$

Hence it follows that

$$\max_{t\in[0,1]} |\Delta x(t)| \leqslant \max_{t,s\in[0,1]} |G(t,s)|\, \|y\|. \tag{12}$$

From (6) we have

$$\max \left| \varepsilon^2 \frac{d}{dt} \left[f(t) \frac{d(\Delta x)}{dt} \right] \right| \leqslant \max_{t\in[0,1]} |\Psi'_u(x_0(t),t)| \cdot \max_{t\in[0,1]} |\Delta x(t)| + \|y\|. \tag{13}$$

Substituting (12) into (13), we obtain

$$\max \left| \varepsilon^2 \frac{d}{dt} \left[f(t) \frac{d(\Delta x)}{dt} \right] \right| \leqslant \theta \|y\|, \tag{14}$$

where

$$\theta = \max_{t\in[0,1]} |\Psi'_u(x_0(t),t)| \cdot \max_{t,s\in[0,1]} |G(t,s)| + 1.$$

Substituting, finally, (12) and (14) into (2), we obtain

$$\|\Delta x\| = (\lambda\theta + \max_{t,s\in[0,1]} |G(t,s)|) \|y\|. \tag{15}$$

From (15) it is clear that

$$\|\Gamma\| \leqslant (\lambda\theta + \max_{t,s\in[0,1]} |G(t,s)|). \tag{16}$$

Estimating $\max_{t,s\in[0,1]} |G(t,s)|$, we have

$$\max_{t,s\in[0,1]} |G(t,s)| \leqslant$$

$$\leqslant M\varepsilon \left| \frac{\exp\left(-\frac{2}{\varepsilon}\int_t^1 \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right) + \exp\left(-\frac{2}{\varepsilon}\int_0^t \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right) - \exp\left(-\frac{2}{\varepsilon}\int_0^1 \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right) - 1}{1 - \exp\left(-\frac{2}{\varepsilon}\int_0^1 \left(\frac{\Psi'_u}{f}\right)^{1/2} d\tau\right)} \right| \leqslant$$

$$\leqslant M\varepsilon, \tag{17}$$

where

$$M = \frac{f(0)}{2} \max_{t\in[0,1]} |(\Psi'_u f)^{-1/2}|.$$

Thus,

$$\|\Gamma\| \leqslant (M\varepsilon + \lambda\theta). \tag{18}$$

Applying theorem 1 (2.XVIII)[(2)] of L. V. Kantorovich, we carry out the estimates

$$1. \qquad [\Gamma(P(x_0))]y = \int_0^1 G(t,s)y_0(s)y(s)\,ds =$$

$$= \left\{\int_0^\varepsilon + \int_\varepsilon^{1-\varepsilon} + \int_{1-\varepsilon}^1\right\}(G(t,s)y_0 y\,ds) \leqslant MN\varepsilon^3(1-2\varepsilon)\|y\| + 2MQ\varepsilon^2\|y\| =$$

$$= [MN\varepsilon^3(1-2\varepsilon) + 2MQ\varepsilon^2]\|y\|.$$

But

$$\|[\Gamma(P(x_0))]y\| \leqslant \|\Gamma(P(x_0))\|\|y\|.$$

Consequently,

$$\|\Gamma(P(x_0))\| \leqslant [MN\varepsilon^3(1-2\varepsilon) + 2MQ\varepsilon^2] = O(\varepsilon^2). \tag{19}$$

Here

$$N = \max_{t\in[0,1]}\left|\frac{d}{dt}\left[f(t)\frac{dx^*(t)}{dt}\right]\right|,$$

$$Q = \max_{t\in[0,1]}\left|\varepsilon^2\frac{d}{dt}\left[f(t)\frac{d\bar{x}_0(t)}{dt}\right] - \Psi(\bar{x}_0(t),t)\right|.$$

2.

$$\|\Gamma P''(x)\| \leq \|\Gamma\| \, \|P''(x)\| \leq (M\varepsilon + \lambda\theta) \max_{t\in[0,1]} |\Psi'_u(x,t)| =$$
$$= K(M\varepsilon + \lambda\theta), \tag{20}$$

where

$$K = \max_{t\in[0,1]} |\Psi'_u(x(t),t)| \quad (x \in \Omega_0).$$

3.

$$\|\Gamma P'(x_0) - I\| = \|\Gamma[P'(x_0) - P'(x)]\| = \left\| \int_x^{x_0} \Gamma P''(x)\, dx \right\| \leq$$
$$\leq K(M\varepsilon + \lambda\theta)\|x_0 - x\| \leq K\delta(M\varepsilon + \lambda\theta). \tag{21}$$

The solvability conditions for the boundary-value problem (1) are represented in the form

$$\bar{h} = \frac{2MQK\varepsilon^2(M\varepsilon + \lambda\theta)}{[1 - K\delta(M\varepsilon + \lambda\theta)]^2} \leq \frac{1}{2}; \qquad K\delta(M\varepsilon + \lambda\theta) < 1. \tag{22}$$

For sufficiently small λ (in absolute value), the solvability conditions (22) are satisfied.

The solution of the boundary-value problem (1), $x^{**}(t)$, exists for

$$r \geq \bar{r}_0 = \frac{1 - \sqrt{1 - 2\bar{h}}}{\bar{h}} \, \frac{O(\varepsilon^2)}{1 - K\delta(M\varepsilon + \lambda\theta)}$$

and is unique for

$$r < \bar{r}_1 = \frac{1 + \sqrt{1 - 2\bar{h}}}{\bar{h}} \, \frac{O(\varepsilon^2)}{1 - K\delta(M\varepsilon + \lambda\theta)}.$$

At the same time, the estimate of convergence of the initial approximation $x_0(t)$ to the exact solution $x^{**}(t)$, according to Theorem 1 (2.XVIII) [(2)] of L. V. Kantorovich, will be

$$\|x^{**} - x_0\| \leq \frac{4MQ\varepsilon^2}{1 - K\delta(M\varepsilon + \lambda\theta)} = O(\varepsilon^2). \tag{23}$$

But it follows from this that, as $\varepsilon \to 0$, the solution of the boundary-value problem (1), $x^{**}(t)$, on the interior interval $[\varepsilon, 1-\varepsilon]$ converges uniformly to the solution $x^*(t)$ of the degenerate equation $\Psi(x(t),t) = 0$ with rate $O(\varepsilon^2)$.

It is easy to see that by the indicated method one can find an approximate solution of boundary-value problems for the equation $\varepsilon^2 y'' = f(x, y, y')$ and give the corresponding estimates.

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Note: Figure translations are in progress. See original paper for figures.