# MATHEMATICS

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## Abstract

## Full Text

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# ON A BASIS IN A SPACE OF ABSTRACT FUNCTIONS

*(Presented by Academician S. L. Sobolev on 15 VII 1960)*

$1^0$. Let us consider simultaneously two Banach spaces: an abstract space $X$ and a function space $B$.

Of the space $X$ we shall require only that it have a basis $\{u_i\}$, whose conjugate functionals we denote by $\{f_i\}$.

We impose a number of conditions on the space $B$:

I. A function $a(t)$ $(0 \leqslant t \leqslant 1)$ belongs to $B$ if and only if there exists a sequence of step functions $\{a_n(t)\}$, converging almost everywhere to $a(t)$ and fundamental in the norm of this space: $\|a_m(t) - a_n(t)\|_B \to 0$ as $m, n \to \infty$. We set that the norm of $a(t) \in B$ is defined by the equality

$$\|a(t)\|_B = \lim_{n\to\infty} \|a_n(t)\|_B.$$

Here it is assumed that the norm of a function equal to zero almost everywhere is zero.

II. If $a(t)$ and $b(t)$ are arbitrary functions from $B$ and $|a(t)| \leqslant |b(t)|$ almost everywhere, then $\|a(t)\|_B \leqslant \|b(t)\|_B$. In particular, $\||a(t)|\|_B = \|a(t)\|_B$.

III. For any step function $a(t)$,

$$\left|\int_0^1 a(t)\,dt\right| \leqslant C\|a(t)\|_B. \tag{1}$$

IV. The space $B$ possesses a basis $\{e_i(t)\}$, whose conjugate functionals we denote by $\{g_i\}$.

For functions of the space $B$ the integral is defined in the usual way. If $a(t) \in B$ and $\{a_n(t)\}$ is a sequence of step functions converging to $a(t)$ almost everywhere and fundamental in the norm of $B$, then

$$\int_0^1 a(t)\,dt = \lim_{n\to\infty} \int_0^1 a_n(t)\,dt.$$

In this case inequality (1) is satisfied for all functions from $B$.

**Theorem 1.** *If a sequence $\{a_n(t)\} \subset B$ converges almost everywhere to a function $a(t)$ and is fundamental in the norm of $B$, then $a(t)$ belongs to $B$ and the sequence $\{a_n(t)\}$ converges to $a(t)$ in the norm of $B$.*

$2^0$. In what follows we shall denote by $\mathfrak{M}$ the set of all step functions defined on the interval $0 \leqslant t \leqslant 1$ with values in $X$.

Denote by $B^X$ the set containing those and only those functions $\varphi(t)$ $(0 \leqslant t \leqslant 1)$ with values in $X$, for each of which there is a sequence $\{\varphi_n(t)\} \subset \mathfrak{M}$, converging to $\varphi(t)$ almost everywhere and fundamental in the sense that

$$\left\|\|\varphi_m(t) - \varphi_n(t)\|_X\right\|_B \to 0 \quad \text{as } m, n \to \infty.$$

We set

$$\|\varphi(t)\|_{B^X} = \lim_{n\to\infty} \left\|\|\varphi_n(t)\|_X\right\|_B.$$

$B^X$ is a linear normed space possessing the properties:

1. If $\varphi(t) \in B^X$, then $\|\varphi(t)\|_X \in B$ and

$$\left\|\|\varphi(t)\|_X\right\|_B = \|\varphi(t)\|_{B^X}.$$

2. If $a(t) \in B$ and $x \in X$, then $a(t)x \in B^X$.
3. If $f \in X^*$ and $\varphi(t) \in B^X$, then $f[\varphi(t)] \in B$.
4. For $\varphi(t)$ from $B^X$ the integral is defined by

$$\int_0^1 \varphi(t)\, dt = \lim_{n\to\infty} \int_0^1 \varphi_n(t)\, dt,$$

where $\{\varphi_n(t)\} \subset \mathfrak{M}$ converges to $\varphi(t)$ almost everywhere and to itself in the norm $B^X$. Moreover, the inequality

$$\left\| \int_0^1 \varphi(t)\, dt \right\|_X \leq \int_0^1 \|\varphi(t)\|_X\, dt$$

is valid.

5. If $f \in X^*$ and $\varphi(t) \in B^X$, then

$$\int_0^1 f[\varphi(t)]\, dt = f\left[\int_0^1 \varphi(t)\, dt\right].$$

6. If $\varphi(t) \in B^X$ and $\{\varphi_n(t)\} \subset \mathfrak{M}$ converges to $\varphi(t)$ almost everywhere and to itself in the norm $B^X$, then $\{\varphi_n(t)\}$ converges to $\varphi(t)$ in the norm $B^X$.

**Lemma 1.** If the sequence $\{\varphi_n(t)\} \subset B^X$ converges almost everywhere to the function $\varphi(t)$ and to itself in the norm $B^X$, then $\varphi(t) \in B^X$ and $\{\varphi_n(t)\}$ converges to $\varphi(t)$ in the norm $B^X$.

**Theorem 2.** The space $B^X$ is complete.

Set $x_{ij}(t) = e_i(t)u_j$. We pose the problem* of proving that the functions $x_{ij}(t)$, arranged in a certain way into a simple sequence, form a basis in the space $B^X$. To solve the problem posed, we shall need one more assumption concerning the space $B$, which will be stated in the next paragraph.

3°. A functional $g \in B^*$ may be regarded as an operator acting in the space $B^X$ and mapping its elements into elements of $X$.

For this purpose we define the value of the operator $g$ on a step function $\varphi(t)$,

$$\varphi(t) = \sum_{i=1}^{m} x_i \chi_{\Delta_i}(t)$$

(where $\Delta_i$ $(i = 1, 2, \dots, m)$ are intervals of constancy of $\varphi(t)$, $x_i \in X$ are its values on these intervals, and $\chi_{\Delta_i}(t)$ is the characteristic function of $\Delta_i$), by putting

$$g\varphi(t) = \sum_{i=1}^{m} x_i g[\chi_{\Delta_i}(t)].$$

Then the operator $g$ will be defined on $\mathfrak{M}$ and will be additive and homogeneous. It will also be bounded, since

$$\|g\varphi(t)\|_X \leq \|g\|_{B^*} \|\varphi(t)\|_{B^X}.$$

Extending $g$ by continuity to the whole space, we obtain a linear operator defined on $B^X$ and mapping it into $X$.

The operator $g$ thus defined has the following two properties:

1) If $a(t) \in B$, $x \in X$, and $g \in B^*$, then $g[a(t)x] = xg[a(t)]$.
2) If $f \in X^*$, $g \in B^*$, and $\varphi(t) \in B^X$, then $f[g\varphi(t)] = g\{f[\varphi(t)]\}$.

Consider the expression

$$K_m(t) = \sum_{i=1}^{m} e_i(t) g_i, \tag{2}$$

which, for fixed $t$, represents an element of $B^*$. In view of the preceding, we may regard it as an operator acting in the space $B^X$ and mapping $B^X$ into $B^X$. For $\varphi(t) \in B^X$,

$$K_m(t)\varphi(\tau) = \sum_{i=1}^{m} x_i e_i(t), \qquad x_i = g_i\varphi(\tau).$$

Now one more condition imposed on $B$ may be formulated as follows:

V. There exists a constant $K > 0$ such that

$$\|K_m(t)\varphi(\tau)\|_{B^X} \leq K\|\varphi(t)\|_{B^X}.$$

4°. **Lemma 2.** For every function $\varphi(t) \in B^X$,

$$\varphi(t) = \sum_{i=1}^{\infty} x_i e_i(t),$$

where $x_i = g_i\varphi(t)$, and the series converges in the norm $B^X$.

* This problem was posed by M. A. Krasnosel' skii at the Voronezh seminar on functional analysis.

**Lemma 3.** For any function $\varphi(t) \in B^X$,

$$\varphi(t) = \sum_{i=1}^{\infty} f_i[\varphi(t)]\, u_i,$$

where the series converges in the norm of the space $B^X$.

**Lemma 4.** The sequence

$$S_n\varphi(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_{ij} x_{ij}(t), \quad \text{where } \xi_{ij} = g_i\{f_j[\varphi(t)]\}, \quad x_{ij}(t) = e_i(t)u_j, \tag{3}$$

converges in the space $B^X$, whatever $\varphi(t) \in B^X$ may be.

5°. Let us now arrange the double sequence $\{x_{ij}(t)\}$ $(i, j = 1, 2, ...)$ into a simple sequence and, correspondingly, the double sum (3) as $n \to \infty$ into a simple series, so that the following condition is satisfied: if the sum of the first $m$ terms of this series is taken, then for $m = n^2$ this sum must coincide with (3). This can be done according to the following scheme: the terms of the sequence $\{x_{ij}(t)\}$ are arranged in groups of $2n - 1$ terms $(n = 1, 2, ...)$, in each of which the largest of the indices $i$ and $j$ is equal to $n$; in each group the terms are arranged in increasing order of the first index, and, when the first indices are equal, in increasing order of the second. We obtain the sequence (4) and the series (5):

$$x_{11},\ x_{12},\ x_{21},\ x_{22},\ x_{13},\ x_{23},\ x_{31},\ x_{32},\ x_{33},\ x_{14}, ... \tag{4}$$

$$\xi_{11}x_{11}(t) + \xi_{12}x_{12}(t) + \xi_{21}x_{21}(t) + \xi_{22}x_{22}(t) + \xi_{13}x_{13}(t) + ... \tag{5}$$

We shall prove that the series (5) converges in the norm of the space $B^X$, whatever the function $\varphi(t) \in B^X$ may be. Since, moreover, the system (4) is complete in the space $B^X$, which follows directly from the preceding conditions, it will thereby be proved that it is a basis in the space $B^X$.

Denote by $s_m\varphi(t)$ the sum of the first $m$ terms of the series (5). Corresponding to the given $m$, choose $n$ such that $n^2 \le m < (n+1)^2$. Then

$$s_m\varphi(t) = S_n\varphi(t) + \omega_n^{(\alpha)}(t), \tag{6}$$

where $S_n\varphi(t)$ is defined by formula (3), and $\omega_n^{(\alpha)}(t)$ is the sum of a certain number $\alpha$ of terms of the series (5) which enter into $S_{n+1}\varphi(t)$ and do not enter into $S_n\varphi(t)$. More precisely, $\omega_n^{(\alpha)}(t)$ is expressed by one of the following two formulas:

$$\omega_n^{(\alpha)}(t) = \begin{cases} \sum_{i=1}^{\alpha} \xi_{i,n+1} x_{i,n+1}(t), & 1 \le \alpha \le n, \\ \sum_{i=1}^{n} \xi_{i,n+1} x_{i,n+1}(t) + \sum_{i=1}^{\alpha-n} \xi_{n+1,i} x_{n+1,i}(t), & n+1 \le \alpha \le 2n. \end{cases} \tag{7,8}$$

In view of Lemma 4 and equality (6), it is sufficient for us to prove that

$$\|\omega_n^{(\alpha)}(t)\|_{B^X} \to 0$$

as $n \to \infty$, uniformly with respect to $\alpha$. The inequality*

$$\|u_n\|_X \, \|f_n\|_{X^*} \le \beta \tag{9}$$

always holds.

Let $\omega_n^{(\alpha)}(t)$ be expressed by formula (7). Then

$$\omega_n^{(\alpha)}(t) = f_{n+1}\left[\sum_{i=1}^{\alpha} x_i e_i(t)\right] u_{n+1},$$

where $x_i = g_i\varphi(\tau)$. Put

$$\varepsilon_\alpha(t) = \varphi(t) - \sum_{i=1}^{\alpha} x_i e_i(t).$$

By Lemma 2 we may assert that, for a given number $\varepsilon > 0$, there is a natural number $\alpha_1(\varepsilon)$ such that for $\alpha \ge \alpha_1$ one has

$$\|\varepsilon_\alpha(t)\|_{B^X} < \varepsilon. \tag{10}$$

Let $\alpha = 1, 2, \dots, \alpha_1 - 1$. Then, by Lemma 3, for sufficiently–

* Its validity for an arbitrary basis was pointed out to me by M. A. Krasnosel' skii.

for sufficiently large $n$,

$$\left\|\omega_n^{(\alpha)}(t)\right\|_{B^X} = \left\|f_{n+1}\left[\sum_{i=1}^{\alpha} x_i e_i(t)\right] u_{n+1}\right\|_{B^X} < \varepsilon.$$

If, however, $\alpha \geqslant \alpha_1$, then, by Lemma 3,

$$\|f_{n+1}[\varphi(t)]u_{n+1}\|_{B^X} < \varepsilon,$$

and, taking into account (9) and (10), we obtain

$$\left\|\omega_n^{(\alpha)}(t)\right\|_{B^X} \leqslant \|f_{n+1}[\varphi(t)]u_{n+1}\|_{B^X} + \|f_{n+1}[\varepsilon_\alpha(t)]u_{n+1}\|_{B^X} < (1+\beta)\varepsilon.$$

Let now the expression (8) hold for $\omega_n^{(\alpha)}(t)$. The first term on the right-hand side of (8) tends to zero by the same considerations as in the preceding case. It remains to prove the convergence to zero of the second term.

We have, by Lemma 2,

$$\left\|\sum_{i=1}^{\alpha-n} \xi_{n+1,i}\, x_{n+1,i}(t)\right\|_{B^X} = \left\|\sum_{i=1}^{\alpha-n} f_i[x_{n+1}e_{n+1}(t)]u_i\right\|_{B^X} \leqslant$$

$$\leqslant M\,\|x_{n+1}e_{n+1}(t)\|_{B^X} \to 0 \quad \text{as } n \to \infty$$

independently of $\alpha$. Consequently, uniformly in $\alpha$,

$$\left\|\omega_n^{(\alpha)}(t)\right\|_{B^X} \to 0 \quad \text{as } n \to \infty,$$

and thus the following has been proved:

**Theorem 3.** *Under the conditions listed above, the system (4) is a basis in the space $B^X$.*

6°. We shall show that the separable Orlicz space $L_M^*$, $E_M$ [(1)], and $L^p$ $(1 \leqslant p < \infty)$ satisfy all the conditions imposed on the space $B$, if in them one considers the basis consisting of the Haar functions. The space $B^X$ for $B = L_M^*$ (or $E_M$) will be denoted by $B_M$. Conditions I–III are verified without difficulty. It remains to prove the validity of condition V.

In the space $B_M$ we have

$$K_m(t)\varphi(\tau) = \int_0^1 N_m(t,\tau)\varphi(\tau)\,d\tau, \qquad N_m(t,\tau) =$$

$$= \sum_{i=1}^{m} e_i(t)e_i(\tau).$$

Therefore

$$\|K_m(t)\varphi(\tau)\|_X \leqslant \int_0^1 |N_m(t,\tau)|\,\|\varphi(\tau)\|_X\,d\tau.$$

Denote by $a_1, a_2, \ldots$ the points of division of the interval $[0,1]$ which enter into the definition of the first $m$ Haar functions $e_1(t), e_2(t), \ldots, e_m(t)$, arranged in increasing order and including the points 0 and 1. The function $N_m(t,\tau)$, for $a_k < t < a_{k+1}$, assumes the following values:

$$N_m(t,\tau) = \frac{1}{a_{k+1} - a_k},$$

$$a_k < \tau < a_{k+1}; \qquad N_m(t,\tau) = 0, \quad \tau < a_k \text{ or } \tau > a_{k+1}.$$

Consequently, for $a_k < t < a_{k+1}$,

$$\int_0^1 |N_m|\,\|\varphi(\tau)\|_X\,d\tau = \frac{1}{a_{k+1} - a_k}\int_{a_k}^{a_{k+1}} \|\varphi(\tau)\|_X\,d\tau.$$

Applying Jensen' s integral inequality, we obtain, for $a_k < t < a_{k+1}$,

$$M\left[\|K_m(t)\varphi(\tau)\|_X\right] \leqslant \frac{1}{a_{k+1} - a_k}\int_{a_k}^{a_{k+1}} M[\|\varphi(\tau)\|_X]\,d\tau.$$

From this inequality follows the inequality

$$\int_0^1 M\left[\|K_m(t)\varphi(\tau)\|_X\right]\,dt \leqslant \int_0^1 M[\|\varphi(t)\|_X]\,dt,$$

from which we obtain

$$\|K_m(t)\varphi(\tau)\|_{B_M} \leqslant \|\varphi(t)\|_{B_M},$$

which proves the validity of condition V in the space $B_M$. The Bochner space [2] and $B_p$ [3] are particular cases of the space $B_M$.

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## REFERENCES

1. M. A. Krasnosel' skii, Ya. B. Rutickii, *Convex Functions and Orlicz Spaces*, Moscow, 1958.

2. E. Hille, *Functional Analysis and Semigroups*, Moscow, 1951.

3. S. L. Sobolev, DAN, **114**, No. 6 (1957).

*Note: Figure translations are in progress. See original paper for figures.*