# MATHEMATICS

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## Abstract

## Full Text

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# ON THE ISOMETRIC IMMERSION IN THE LARGE OF A TWO-DIMENSIONAL RIEMANNIAN MANIFOLD IN A THREE-DIMENSIONAL ONE

Let $R$ be a three-dimensional Riemannian space and $M$ a two-dimensional Riemannian manifold homeomorphic to a sphere. The problem considered in the present note consists in constructing a regular surface $F$, isometric to $M$, in the space $R$. This problem was posed by the author in paper (1) and solved there under the condition that the Gaussian curvature of the immersed manifold $M$ is greater than a certain constant $k^*$, depending on the curvature of the space $R$.

$$k^* = \max_R \max_P \{k_1,\ 3(k_1 - k_2)\},$$

where $k_1$ and $k_2$ are, respectively, the greatest and the least of the curvatures of the space in two-dimensional planes passing through the point $P$. We have succeeded in substantially improving this result for the case of spaces of non-positive curvature, so that the theorem on the isometric immersion of $M$ in $R$ for spaces of this kind has assumed, in a certain sense, a definitive form.

**Theorem 1.** Let $R$ be a complete three-dimensional Riemannian space with curvature in two-dimensional planes not exceeding $c_0 \leqslant 0$; let $M$ be a closed two-dimensional manifold, homeomorphic to a sphere, with Gaussian curvature everywhere greater than $c_0$.

Then, if the metrics of the space $R$ and of the manifold $M$ are sufficiently regular, the manifold $M$ is isometrically immersed in $R$ in the form of a certain regular surface $F$.

Moreover, this immersion can be carried out so that a given two-dimensional element $\alpha$ of the manifold $M$ (a point and a pencil of directions at it) coincides with a given two-dimensional element $\alpha'$ of the space $R$ isometric to $\alpha$, and the surface $F$ lies on a prescribed side of the plane of the element $\alpha'$.

If the metrics of the space $R$ and of the manifold $M$ are $k$-times differentiable ($k \geq 6$), then the surface $F$ is differentiable at least $k-2$ times. If the metrics of the space and of the manifold are analytic, then the surface $F$ is analytic.

The immersion of $M$ in $R$ determined by the indicated conditions is unique.

The assertion of uniqueness of the surface $F$, fixed by a two-dimensional element in Theorem 1, is supplemented by the following theorem on isometric transformations of a free surface.

**Theorem 2.** Suppose that in a complete Riemannian space with regular metric and curvature everywhere not exceeding $c_0 \leqslant 0$, two regular, identically oriented surfaces homeomorphic to a sphere are given, with Gaussian curvature greater than $c_0$. Then there exists a continuous bending of one surface into the other, i.e. a continuous deformation preserving the metric of the surface.

The proof of Theorem 1, just as in work ([1]), consists of three parts. First, one proves the existence of a continuous family of manifolds $M_t$ containing the given manifold $M$ and a manifold $M_0$ that is certainly immersible. Second, one proves that if the manifold $M_t$ is immersible, then manifolds of the family close to it are also immersible. Finally, one proves that if each manifold $M_{t_n}$ of the family is immersible and $t_n \to t^*$, then the manifold $M_{t^*}$ is immersible.

Concerning the proofs of the assertions contained in these three parts, we note the following. The proof of the assertion that manifolds $M_t$ close to an immersible one are immersible, given in work ([1]), essentially relies only on the fact that the Gaussian curvature of the manifolds $M_t$ is greater than the curvature of the space. Therefore, if we can construct the family of manifolds $M_t$ in such a way that their Gaussian curvatures are greater than $c_0$, then the assertion of the second part may be considered proved.

The proof of the assertion contained in the third part, on the immersion of the manifold $M_{t^*}$, essentially relies only on the possibility of establishing a priori estimates for the normal curvatures of the surfaces $F_t$ isometric to the manifolds $M_t$. The immersion condition $k > k^*$ in work ([1]) was due to the fact that only under this condition were estimates obtained for the normal curvatures of the surfaces $F_t$. In the case of spaces of nonpositive curvature we can now weaken the condition for obtaining a priori estimates to the requirement of positivity of the external curvature of the surface. Thus, the assertion of the third part may be considered proved if all manifolds $M_t$ of the family have Gaussian curvatures greater than $c_0$.

It remains to construct a continuous family of manifolds $M_t$ containing the manifold $M$ and the manifold $M_0$, which is certainly immersible, in such a way that the Gaussian curvatures of the manifolds are greater than $c_0$. Such a family is not difficult to construct. Take in the space $R$ a small sphere $\omega$ tangent to the two-dimensional element $a'$ at its center. The external curvature of $\omega$ is positive; consequently, the Gaussian curvature is greater than $c_0$. We now construct in Lobachevsky space with curvature $c_0$ (in the case $c_0 = 0$, in Euclidean space) surfaces $\bar{F}$ and $\bar{\omega}$, isometric respectively to $M$ and $\omega$. The construction of such surfaces is guaranteed by A. D. Aleksandrov' s theorem on the realization of an abstractly prescribed metric by a convex surface in Lobachevsky space (2). The surfaces $\bar{F}$ and $\bar{\omega}$ are regular (3). Map Lobachevsky space with the surfaces $\bar{F}$ and $\bar{\omega}$ geodesically onto the interior of a Euclidean ball and construct a linear family of surfaces $\widetilde{F}_t$ joining the images of the surfaces $\bar{F}$ and $\bar{\omega}$ inside the ball. The surfaces $\bar{F}_t$ of Lobachevsky space corresponding to the surfaces $\widetilde{F}_t$ are strictly convex, their external curvatures are positive, and consequently their Gaussian curvatures are greater than $c_0$. The family of surfaces $\bar{F}_t$ with their natural metric is the required family of manifolds $M_t$.

What is essentially new in the proof of Theorem 1 in comparison with the proof in work (1) is the establishment of a priori estimates of normal curvatures under the sole condition of positivity of the external curvature of the surface. We can give some idea of the method for obtaining the estimates if we say that it very remotely resembles the method of obtaining estimates for convex surfaces in Lobachevsky space (3).

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## CITED LITERATURE

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2. A. D. Aleksandrov, *Intrinsic Geometry of Convex Surfaces*, 1948.

3. A. V. Pogorelov, DAN, **137**, No. 1 (1961).

*Note: Figure translations are in progress. See original paper for figures.*