ON NONUNIQUE "SMEARING" OF DISCONTINUITIES IN SOLUTIONS OF QUASILINEAR SYSTEMS

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Abstract

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ON NONUNIQUE "SMEARING" OF DISCONTINUITIES IN SOLUTIONS OF QUASILINEAR SYSTEMS

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This note will deal with the theory of ordinary differential equations describing the "smearing" of discontinuities in quasilinear hyperbolic systems. In order to carry out such smearing, we must replace the zeros on the right-hand side of the system

$$\frac{\partial F_i(q_1, q_2, \dots, q_n)}{\partial t} + \frac{\partial G_i(q_1, q_2, \dots, q_n)}{\partial x} = 0 \tag{1}$$

by viscous terms

$$\frac{\partial}{\partial x}\left(\varepsilon \sum b_{ik} \frac{\partial q_k}{\partial x}\right).$$

Ordinary equations are obtained if one seeks solutions of the form

$$q_j = q_j\left(\frac{t - \alpha x}{\varepsilon}\right) = q_j(\tau).$$

I. M. Gelfand posed in [1] the problem of studying these equations. We shall show that there may exist quite reasonable $\|b_{ik}\|$ for which the solution describing a smeared discontinuity (shock wave) is not unique.

Consider the system

$$\frac{\partial L_{q_i}}{\partial t} + \frac{\partial L^1_{q_i}}{\partial x} = \frac{\partial}{\partial x}\left(\varepsilon \sum b_{ik} \frac{\partial q_k}{\partial x}\right), \tag{2}$$

where

$$L = q_1^2 + 3q_2^2 + 5q_3^2 + 2e^{q_1} + 4e^{q_2} + 6e^{q_3},$$

Fig. 1

Figure 1: Fig. 1

$$L^1 = q_1^2 + q_2^2 + q_3^2 + e^{q_1} + e^{q_2} + e^{q_3}.$$

It is easy to verify that this system, for $\varepsilon = 0$, is hyperbolic. To ensure the evolutionary character (in the linear approximation) of system (2) with $\varepsilon > 0$, it is sufficient to require the positive definiteness of $\|b_{ik}\|$. For (2) we shall seek solutions of the form $q_i = q_i(\tau)$, $\left(\tau = \dfrac{t - \alpha x}{\varepsilon}\right)$. The equations for $q_i(\tau)$, using the first integrals available for them, can be reduced to the form:

$$\Lambda_{q_i} = -\alpha^2 D_{\sigma_i},$$

$$dq_i = -\sigma_i \, d\tau.$$

Here

$$\Lambda = L - \alpha L^1 - A_1 q_1 - A_2 q_2 - A_3 q_3; \qquad D = \frac{1}{2} \sum b_{ik} \sigma_i \sigma_k;$$

A_1, A_2, A_3 are constants of integration.

The trajectories defined in the space (q_1, q_2, q_3) by equations (3) are orthogonal, in the sense of the metric specified by the form D, to the level surfaces of the function Λ. Motion along these trajectories occurs in

in the direction of increase of Λ as τ increases. The smeared discontinuities correspond to $q_i(\tau)$ tending to finite limits as $\tau \to \pm\infty$. It is not difficult to see that these limits are stationary points of the function $\Lambda(q_1, q_2, q_3)$.

In our example we set $\alpha = 7/2,\ A_1 = -8/2,\ A_2 = 0,\ A_3 = 1/2$. The stationary points of Λ have coordinates $q_1 = 0,\ q_3 = 0$, while q_2 is found from the equation $1/2 e^{q_2} - q_2 - 1 = 0$, which has two roots (the larger q_2 corresponds to the smaller value of Λ).

It is easy to see that the surfaces $\Lambda = \text{const}$, for Λ greater or less than both critical values, are homeomorphic to a plane.

Since the topological structure of the level surfaces can change only when Λ passes through critical values, it will be the same for all Λ lying between the critical values. The structure of a typical such surface is shown in Fig. 1.

Fig. 1

Fig. 2

Figure 2: Fig. 2

The points A and B are critical for Λ, with A situated "inside the handle" (for $\Lambda < \Lambda_0$), and B "outside the handle" (for $\Lambda > \Lambda_0$). The structure of the level surfaces near the point A is shown in Fig. 2. Lines orthogonal to the level surfaces and issuing from A in the direction of increase of Λ intersect, on each of the surfaces with $\Lambda_B > \Lambda > \Lambda_A$, a one-dimensional closed cycle Z_A, enclosing the handle as shown in the figures.

Fig. 2

The structure of the point B is analogous to the structure of A. The orthogonal trajectories of interest to us, issuing from B, intersect on $\Lambda = \text{const}$ ($\Lambda_B > \Lambda > \Lambda_A$) a cycle Z_B, situated as shown in Fig. 1.

It is clear that Z_A and Z_B intersect; i.e., there exists at least one trajectory connecting the points A and B. However, this trajectory need not be unique.

Indeed, by choosing Λ_1 and Λ_2 ($\Lambda_B > \Lambda_1 > \Lambda_2 > \Lambda_A$) and changing the metric D in the region $\Lambda_1 > \Lambda > \Lambda_2$, one can ensure that the cycle Z_A on $\Lambda = \Lambda_2$ and the cycle Z_B on $\Lambda = \Lambda_1$ are connected by more than one trajectory orthogonal to the surfaces $\Lambda = \text{const}$. The construction of such a metric must begin with drawing the trajectories themselves, and then constructing D along the trajectories so as to satisfy the orthogonality condition.

After this, D may be extended to the whole region $\Lambda_1 > \Lambda > \Lambda_2$ quite arbitrarily.

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REFERENCES

1. I. M. Gelfand, *UMN*, **14**, no. 2 (86), 87 (1959).

Note: Figure translations are in progress. See original paper for figures.