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Abstract

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DIVERGENT SERIES FROM L^2 WITH RESPECT TO COMPLETE SYSTEMS

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§ 1. The main content of the note is the following theorem:

Theorem 1. Let $\{\varphi_n(x)\}$ be an arbitrary complete orthogonal system of functions in $L^2[0,1]$. Then there exists a function $f(x) \in L^2[0,1]$ whose Fourier series

$$\sum_{n=1}^{\infty} b_n \varphi_n(x) \tag{1}$$

after a certain rearrangement of its terms diverges almost everywhere (a.e.) on $[0,1]$.

As an immediate consequence of this theorem we obtain a negative answer to the problem of the existence of a complete orthonormal system of unconditional convergence.

(As is known, an orthonormal system $\{\varphi_n(x)\}$ is called a system of convergence if from the condition

$$\sum_{n=1}^{\infty} b_n^2 < \infty \tag{2}$$

there follows the convergence of the series (1) a.e. on $[0,1]$. If the system $\{\varphi_n(x)\}$ remains a system of convergence under any rearrangement of its elements, then it is called a system of unconditional convergence. A classical example of such a system is the Rademacher system. However, it, as well as all other known systems of unconditional convergence, is not complete. In this connection there naturally arose the question (([6], p. 452): does there exist a complete system of unconditional convergence? The answer to this question follows immediately from Theorem 1. Namely, the following is true:

Theorem 2. There exists no complete orthonormal system of unconditional convergence.

The essential meaning of the results stated is that any complete orthonormal system can be enumerated in such an order $\{\varphi_n(x)\}$ that the properties of this system will be sufficiently bad; namely, with respect to it there will exist divergent series from L^2.

It should be noted that, for the trigonometric system, Theorem 1 was first formulated by A. N. Kolmogorov in 1927 in his joint work with D. E. Men' shov ([1]). Recently a paper by Zagorskii ([3]) appeared, containing a scheme of the proof of Kolmogorov' s theorem. An analysis by P. L. Ul' yanov of the method proposed by Zagorskii showed that it is also applicable to some other systems. Roughly speaking, the condition for applicability of Zagorskii' s method is a certain "symmetry" of the functions of the system. Along this path P. L. Ul' yanov ([7]) extended the assertion of A. N. Kolmogorov' s theorem to the Walsh and Haar systems. In the present work we prove this assertion for any complete orthogonal system.*

§ 2. We proceed to the proof of Theorem 1.

* In the same direction results have been obtained by P. L. Ul' yanov ([8]), whose paper is published in the present issue.

I. Let us agree on the following definition. Let a natural sequence $\Omega = \{s_1, s_2, \dots, s_k, \dots\}$, $s_{k+1} > s_k$, $k = 1, 2, \dots$, be fixed. We shall say that an orthonormal system $\{\varphi_n(x)\}$ is an Ω-unconditional basis if it follows from condition (2) that the series

$$\sum_{k=1}^{\infty} \sigma_k(x), \qquad \text{where} \quad \sigma_k(x) = \sum_{n=s_k+1}^{s_{k+1}} b_n \varphi_n(x), \tag{3}$$

converges under any permutation of its terms on a set of positive measure (depending on the permutation).

The results of the present paragraph may be formulated as a lemma:

Lemma 1. *If there exists a complete orthonormal system $\{\psi_\nu(x)\}$ which is an Ω-unconditional basis (for some $\Omega = \{s_k\}$), then every orthonormal system $\{\varphi_n(x)\}$ is an Ω'-unconditional basis for some $\Omega' = \{m_i\}$ (depending on $\{\varphi_n\}$).*

For the proof of the lemma we use an idea of Marcinkiewicz in the form in which it is presented in ([2]), pp. 352–357. Namely, expanding $\varphi_i(x)$ in Fourier series with respect to the system $\{\psi_\nu(x)\}$ ($\varphi_i(x) \sim \sum a^i_\nu \psi_\nu(x)$), we construct sequences $v(i)$, n_k, $u(i)$ as indicated in ([2]). In doing so, without loss of generality, we may assume that $v(i)$ is a subsequence of Ω. Next, we choose from n_k a subsequence m_i so that $u(m_{i+1}) > v(m_i)$. The sequence $\Omega' = \{m_i\}$ is the desired one. To prove this, we represent, as in ([2]), the series (1) as the sum of three series. Then, by exactly the same estimates as in ([2]), we make sure that the first and third of these series converge absolutely. As for the second series, grouping in it the terms with numbers from $s_k + 1$ to s_{k+1} and denoting this sum by $\sigma_k(x)$, by analogy with (3), we obtain the series $\sum \sigma_k(x)$, which under any permutation converges on a set of positive measure.

The proof of the last assertion is also carried over from ([3]), with the sole difference that, instead of referring to Kolmogorov' s theorem on trigonometric series, in our case one must refer to the fact that the system $\{\psi_\nu(x)\}$, by assumption, is an Ω-unconditional basis. Thus the lemma will be completely proved.

II. The result of the present paragraph may be considered independently of everything else. Here one curious property of the Haar system is clarified. We shall denote the Haar system (see ([2]), p. 57), numbered in the usual order, by $\{\chi_\nu(x)\}$, $\nu = 0, 1, \ldots$. Let a strictly monotonically increasing natural sequence $\Lambda = \{i_1, i_2, \ldots\}$ be fixed. We shall now define the corresponding sequence $F(\Lambda)$ of functions $\{f_1(x), f_2(x), \ldots\}$. Namely, set

$$f_1(x) = 2^{-i_1/2} \sum_{\nu=2^{i_1}}^{2^{i_1+1}-1} \chi_\nu(x).$$

Let now the functions $f_1, \ldots, f_{k-1}$ already be defined. Denote by $E_n^{(1)}(\Lambda)$ the set of points of the interval $[0,1]$ where $f_n(x) > 0$, and by $E_n^{(2)}(\Lambda)$ the set where $f_n(x) < 0$. By $\Phi_n^{(1)}(\Lambda, x)$ and $\Phi_n^{(2)}(\Lambda, x)$ we shall denote respectively the characteristic functions of these sets. Put

$$f_k(x) = 2^{\frac{1}{2}([\log_2 k] - i_k)}\, \Phi_{[k/2]}^{(r)}(\Lambda, x) \sum_{\nu=2^{i_k}}^{2^{i_k+1}-1} \chi_\nu(x), \tag{4}$$

where $r = 1$ if k is even, and $r = 2$ if k is odd.

Lemma 2. *Let an arbitrary increasing sequence of natural numbers Λ be given, and let the sequence*

measurability of the functions $F(\Lambda)$. Then for every n there exists a one-to-one* isometric mapping U_n of the interval $[0,1]$ onto itself such that

$$f_k(x) = \chi_k(U_n x), \qquad k = 1, 2, \ldots, 2^n - 1. \tag{5}$$

Proof. The following assertions follow immediately from the definitions given above.

1. For any k,
$$f_k(x) = (-1)^{r+1} \cdot 2^{\frac{1}{2}[\log_2 k]},$$
if $x \in E_k^{(r)}(\Lambda)$, and $f_k(x) = 0$ at the remaining points of the interval $[0,1]$.

2. The set $E_n^{(r)}(\Lambda)$, for arbitrary Λ, n, and r, consists of a finite number of nonintersecting intervals.

3. $\operatorname{mes} E_k^{(r)} = \mu_k$ depends only on k and does not depend on Λ or r (this fact is easily verified by induction).

We shall agree to say that the collection of sets $E_k^{(r)}(\Lambda)$ $(2^{n-1} \leq k \leq 2^n - 1;\ r = 1, 2)$ are sets of rank n.

It is not difficult to verify the properties:

4. For any fixed Λ and n, the sets of rank n do not intersect and, in their sum, give almost the whole interval $[0, 1]$.
5. For any fixed Λ, every set of rank less than n consists of a finite number of sets of rank n.

Now let Λ, Λ_1, and n be fixed. Then, by properties 2 and 3, each set $E_k^{(r)}(\Lambda)$ of rank n can be mapped one-to-one and isometrically (by means of a piecewise-linear function) onto the corresponding set $E_k^{(r)}(\Lambda_1)$. Combining these mappings, by virtue of 4, we obtain a one-to-one, isometric mapping of the interval $[0, 1]$ onto itself, under which, for any $k \leq 2^n - 1$ and $r \leq 2$, the set $E_k^{(r)}(\Lambda)$ is mapped one-to-one and isometrically onto $E_k^{(r)}(\Lambda_1)$. The latter follows from property 5. By virtue of property 1, we are convinced that

$$f_k(x) = f_k^{(1)}(U_n x), \qquad k \leq 2^n - 1 \quad (\text{where } \{f_k\} = F(\Lambda),\ \{f_k^{(1)}\} = F(\Lambda_1)).$$

To complete the proof of the lemma it remains to put $\Lambda_1 = \{1, 2, ...\}$ and to note that $F(\Lambda_1)$ in this case is the Haar system: $f_k^{(1)}(x) = \chi_k(x)$, $k = 1, 2, \ldots$. Thus, Lemma 2 is proved.

From Lemma 2, with the aid of the usual technique (or by referring to the lemma in our note ([4])), we obtain the following assertion:

Lemma 3. Let the series $\sum a_k \chi_k(x)$, under some rearrangement of its terms, diverge a.e. on $[0, 1]$. Then for any Λ the series $\sum a_k f_k(x)$ diverges a.e. on $[0, 1]$ under the same rearrangement of the terms (here $\{f_k\} = F(\Lambda)$).

III. **Lemma 4.** The Haar system is not an Ω-unconditional basis for any Ω.

Proof. In the proof we shall use a result of P. L. Ulyanov ([7]) on the existence of a Fourier series from L^2

$$\sum_{k=1}^{\infty} a_k \chi_k(x) \tag{6}$$

with respect to the Haar system, which diverges a.e. on $[0, 1]$ under some rearrangement of its terms. Thus, let $\Omega = \{s_k\}$ be fixed. We shall show that the Haar system is not an Ω-unconditional basis. Obviously, without restricting generality, we may assume Ω to be so sparse that for every k there is an i_k such that

$$s_k < 2^{i_k} + 1 < 2^{i_k+1} < s_{k+1}.$$

Assign to each k a unique number i_k so that the written inequality is satisfied. Obviously, $i_{k+1} > i_k$. Thus we define–

* Here and below in the proof we neglect a finite set of points–the ends of intervals of constancy of the functions f_k and χ_k, $k < 2^n$.

the sequence $\Lambda = \{i_k\}$, and with it also $F(\Lambda) = \{f_1(x), f_2(x), ...\}$. Let $s_k \leqslant \nu < s_{k+1}$. Denote by Δ_ν the set on which $\chi_\nu(x)$ is different from zero. Suppose, further, that $2^{i_k} \leqslant \nu \leqslant 2^{i_{k+1}} - 1$. From (4) and property 5 of the preceding paragraph it follows that in this case the set $E_k^{(1)}(\Lambda) \cup E_k^{(2)}(\Lambda)$ either contains Δ_ν or does not intersect it. If the first case occurs, put

$$b_\nu = 2^{\frac{1}{2}([\log_2 k] - i_k)} a_k.$$

For all remaining ν from the half-interval $(s_k, s_{k+1}]$ put $b_\nu = 0$. Thus the sequence of coefficients $\{b_\nu\}$, $\nu = 1, 2, ...$, is completely defined. Consider the series $\sum b_\nu \chi_\nu(x)$. Denote the sum of the segment of this series from $s_k + 1$ to s_{k+1} by $\sigma_k(x)$. From (4) it is immediately clear that $\sigma_k(x) = a_k f_k(x)$. Taking into account the divergence of series (6) under a rearrangement, by Lemma 3 we obtain that the series $\sum \sigma_k(x)$, under some rearrangement of its terms, diverges a.e. on $[0, 1]$. Finally, note that from Lemma 2 it follows that the functions $f_k(x)$ are orthonormal in $L^2[0, 1]$. Taking into account that series (6) converges in L^2, we obtain that the series $\sum \sigma_k(x)$, and hence also (1), is a series from L^2. Consequently, condition (2) is satisfied. Lemma 4 is proved.

Comparing the assertions of Lemmas 1 and 4, we obtain that for no Ω does there exist a complete orthonormal system that is an Ω-unconditional basis. In particular, for $\Omega = \{1, 2, ...\}$ we obtain Theorem 1. Thus Theorem 1, and with it also Theorem 2, are completely proved.

§ 3. From Theorem 1, if one takes into account the results of S. Kaczmarz ([2]), p. 207, we immediately obtain the following assertion:

Theorem 3. *In any complete orthonormal system $\{\varphi_n(x)\}$ one can so change the order of the elements that the Lebesgue functions of the resulting system will be unbounded a.e. on $[0, 1]$.*

At the same time, using the method of our paper ([5]), one can obtain the theorem:

Theorem 4. *For any sequence $\omega(n) \to \infty$ there exists a complete orthonormal system $\{\varphi_n(x)\}$ such that, under any rearrangement of its elements, the Lebesgue functions of the resulting system satisfy the condition $L_n(x) = o(\omega(n))$.*

Theorems 3 and 4 give complete information on the behavior of the Lebesgue functions of rearrangements of complete systems. Let us note one more consequence of Theorems 1 and 4.

Theorem 5. *In order that a given positive sequence* $\omega(n)$ *be a Weyl multiplier for any rearrangement of some complete orthonormal system* $\{\varphi_n(x)\}$*, it is necessary and sufficient that* $\omega(n) \to \infty$*.*

Remark 1. Theorem 1 can be strengthened by replacing in it the divergence of series (1) after a rearrangement by nonsummability by any prescribed method T (T^*). This fact follows directly from Theorem 1 if one applies to it Theorem 5 of our paper ([4]).

Remark 2. The question of the possibility of strengthening Theorem 1 in the sense of replacing the condition $f(x) \in L^2$ by the condition of continuity (or at least boundedness) of the function $f(x)$ remains open.*

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* *Note added in proof.* At present we have obtained a positive answer to this question.

Note: Figure translations are in progress. See original paper for figures.