MATHEMATICS

A. V. KIBENKO

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. V. KIBENKO

ON THE THEORY OF ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

(Presented by Academician P. S. Aleksandrov, October 1, 1960)

1. We study the differential equation

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

in a real Banach space E. For equation (1) new uniqueness theorems are proposed for the solution of the Cauchy problem, i.e., the problem of finding a solution satisfying the prescribed initial condition

$$x(t_0) = x_0. \tag{2}$$

Conditions are considered for the nonlocal continuability of all solutions of equation (1). As is known (^{1}), uniqueness theorems and nonlocal continuability theorems have a common nature; usually these theorems follow from estimates of the values of certain functionals on integral curves. The functionals considered below are constructed by combining constructions of P. Conti (^{2}), O. Borůvka (^{3}), M. A. Krasnosel' skii and S. G. Krein (^{1}), and A. I. Perov (^{4}).

2. Below, $\varphi(t, u, v)$ denotes a nonlinear functional, continuous for

$$t \in (t_0, t_0 + a], \qquad \|u - x_0\| \le b, \qquad \|v - x_0\| \le b,$$

equal to zero for $u = v$, positive for $u \ne v$, and satisfying, for any two solutions $u(t)$ and $v(t)$ of problem (1)–(2), the condition

$$\lim_{t \to t_0 + 0} \varphi[t, u(t), v(t)] = 0. \tag{3}$$

Concerning the smoothness of the functional $\varphi(t, u, v)$, we shall assume that either it is strongly Fréchet differentiable with respect to the collection of variables, or the following less restrictive condition of semidifferentiability is satisfied (cf. (^{4})):

$$\varphi(t+\Delta t, u+\Delta u, v+\Delta v) - \varphi(t,u,v) \le D_1(t,u,v)(\Delta t) + D_2(t,u,v)(\Delta u) + D_3(t,u,v)(\Delta v) + O\left(|\Delta t| + \|\Delta u\| + \|\Delta v\|\right), \tag{4}$$

where the generally nonlinear functionals D_1, D_2, D_3 are semihomogeneous; for $\alpha > 0$,

$$\alpha D_i(t,u,v)(h) \le D_i(t,u,v)(\alpha h),$$

the functional D_1 is continuous, while D_2 and D_3 are upper semicontinuous:

$$\lim_{\|h_n - h\| \to 0} D_i(t,u,v)(h_n) \le D_i(t,u,v)(h_0) \qquad (i = 2,3).$$

If $\varphi(t,u,v)$ is Fréchet differentiable, then the functionals D_i coincide with the gradients of the functional $\varphi(t,u,v)$ with respect to the corresponding variables.

Let us emphasize that the functional $\varphi(t,u,v)$ is not defined for $t = t_0$; its values may increase without bound as $t \to t_0$ and $\|u - v\| \to 0$. Therefore condition (3) is an essential restriction, which, however, is checked in a number of cases without special difficulty (see, in this connection, (^{3}, ^{7})).

In what follows we shall assume that a continuous functional $\Phi(t,u,z)$ of three variables is given,

$$t \in [t_0, t_0 + a], \qquad \|u - x_0\| \le b, \qquad 0 \le z < \infty.$$

The operator $f(t,x)$ is assumed to be defined for $t \in [t_0, t_0 + a]$, $\|x - x_0\| \le b$.

The basic restriction imposed on the right-hand side of equation (1) has the form

$$\begin{gathered} D_1(t,u,v)(1) + D_2(t,u,v)[f(t,u)] + \\ + D_3(t,u,v)[f(t,v)] \le \Phi[t, u\varphi(t,u,v)]. \end{gathered} \tag{5}$$

O. Borůvka (^{3}) considered condition (5) in establishing uniqueness theorems for a scalar equation. The passage to the general case can be carried out according to the usual scheme (^{1}, ^{4}).

Theorem 1. *Suppose condition (5) is fulfilled. Suppose that, for any solution $u(t)$ of problem (1)–(2), all solutions of the problems*

$$\frac{dz}{dt} = \Phi(t, u(t), z), \qquad z(t_0) = 0$$

are nonpositive ($z(t) \leq 0$) *on some interval* $[t_0, t_0 + a_0]$. *Then the solution of problem (1)–(2) is unique on the interval* $[t_0, t_0 + a]$ *(if this solution exists).*

We indicate one modification of Theorem 1, which is based on the idea of the paper by Pliss and Vazhevsky (^{5}) and at the same time is a strengthening of their results.

Let the domain H in the (t, z)-plane be singled out by the inequalities

$$t_0 < t \leq t_0 + a, \qquad 0 \leq z < \infty.$$

We shall say that a function $\Phi(t, z)$, $(t, z \in H)$, possesses property P, if, for $z > 0$, for each real $s \in (t_0, t_0 + a]$ there exist a positive number $\varepsilon > 0$ and a sequence $z_n(t)$ of positive solutions of the equation

$$\frac{dz}{dt} = \Phi(t, z),$$

such that

$$\lim_{n \to \infty} z_n(t) = 0 \qquad (s \leq t \leq s + \varepsilon).$$

Theorem 2. *Suppose condition (5) is fulfilled in the strengthened form*

$$\begin{aligned} D_1(t, u, v)(1) + D_2(t, u, v)[f(t, u)] + \\ + D_3(t, u, v)[f(t, v)] < \Phi[t, \varphi(t, u, v)]. \end{aligned} \tag{6}$$

Suppose the function $\Phi(t, z)$ *possesses property* P. *Then the solution of problem (1)–(2) is unique on the interval* $[t_0, t_0 + a]$ *(if this solution exists).*

3. Let now, for equation (1), the local existence theorem be valid for arbitrary initial conditions. Suppose that for every positive $\alpha > 0$

$$\sup_{0 \leq t \leq a, \ \|x\| \leq a} f(t, x) < \infty.$$

The continuous functional $\varphi(t, u)$ considered in this item ($t \in (0, \infty)$, $u \in E$) and the function $\Phi(t, z)$ ($t \leq z < \infty$) are analogues of the functionals $\varphi(t, u, v)$ and $\Phi(t, u, z)$ used above. It is assumed that for every $a > 0$

$$\lim_{\|u\| \to \infty} \sup_{0 \leq t \leq a} \varphi(t, u) = \infty.$$

Theorem 3. *Suppose the upper solution of the problem*

$$\frac{dz}{dt} = \Phi(t, z), \qquad z(0) = 0 \tag{7}$$

is defined for all $t > 0$. *Suppose that the condition (analogous to inequalities (5) and (6)) is satisfied*

$$D_1(t, u)(1) + D_2(t, u)[f(t, u)] \leq \Phi[t, \varphi(t, u)]. \tag{8}$$

Then it follows from the validity of the local existence theorem for equation (1) that every solution of this equation is continuable up to $t = \infty$.

If it is assumed that all solutions of equation (7) are bounded, the function $\Phi(t, z)$ does not decrease with respect to z, and for each $R > 0$

$$\sup_{t \geq 0,\ \|u\| < R} \varphi(t, u) < \infty,$$

then, under the conditions of Theorem 3, all solutions of equation (1) will be bounded in norm (cf. (6)).

The proof of Theorem 3 uses certain considerations from the paper of M. A. Krasnosel'skii and S. G. Krein (^{1}), in which generalizations of Wintner's original theorems on the nonlocal continuability of solutions were given. Let us note that, for finite systems of scalar equations, nonlocal continuability of solutions under conditions analogous to Theorem 3 was proved by R. Conti (^{2}).

4. The concretization of Theorems 1-3 for integro-differential equations and for normal infinite systems of ordinary differential equations of the first order is carried out according to the usual schemes (see, for example, (^{1},^{4})).

The author takes the opportunity to express his sincere gratitude to M. A. Krasnosel' skii and A. I. Perov for posing the problem, for their attention, and for their advice.

Voronezh State
Pedagogical Institute

Received
30 IX 1960

REFERENCES

1. M. A. **Krasnosel' skii**, S. G. **Krein**, Tr. seminara po funkcional' n. analizu, Voronezh. gos. univ., v. 2 (1956).

2. R. **Conti**, Boll. Unione Mat. Ital., 11, No. 4 (1956).

3. O. **Borůvka**, Acta Fac. Rerum Natur. Univ. Comenianae Math., 1, No. 4-6 (1956).

4. P. A. **Perov**, DAN, 120, No. 4 (1958).

5. A. **Pliś**, T. **Waźewski**, Bull. Akad. Polon. Sci., Ser. Sci. Math., Astr. et Phys., 6, No. 3 (1958).

6. Z. **Szmydt**, Ann. Polon. Math., No. 2 (1955).

7. E. A. **Coddington**, N. **Levinson**, Theory of Ordinary Differential Equations, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.