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Abstract

Full Text

THEORY OF ELASTICITY

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DYNAMICS OF ELASTIC-VISCOUS SHELLS AND PLATES

A general theory of vibrations of elastic-viscous shells is presented below. The shell material is assumed to be isotropic, homogeneous, and to obey a linear relation between three tensors—stress, stress rate, and strain rate. For the shell the Love-Kirchhoff hypotheses are valid. The shell is shallow; displacements of its middle surface are assumed small. A system of differential equations for the problem is obtained, which is solved for the case of a circular cylindrical shell streamlined by a supersonic gas flow along the generator.

1. Elastic-viscous shallow shells. The equilibrium equations for a small element of the shell have the form

$$\frac{\partial N_1}{\partial x}+\frac{\partial T_2}{\partial y}+X=0,\qquad \frac{\partial N_2}{\partial y}+\frac{\partial T_1}{\partial x}+Y=0,\qquad \frac{N_1}{R_1}+\frac{N_2}{R_2}+\frac{\partial Q_1}{\partial x}+\frac{\partial Q_2}{\partial y}+Z=0,$$

$$\frac{\partial H_1}{\partial x}+\frac{\partial M_2}{\partial y}-Q_2=0,\qquad \frac{\partial H_2}{\partial y}+\frac{\partial M_1}{\partial x}-Q_1=0; \qquad (1,1)$$

here $N_1, N_2, T_1=T_2=T,\ Q_1, Q_2,\ M_1, M_2, H_1=H_2=H$ are specific forces and moments; X, Y, Z are the components of the external surface load respectively along the orthogonal axes x, y, z; R_1, R_2 are the principal radii of curvature. The coordinate system coincides with the principal directions on the middle surface.

If $\sigma_1(z),\ \sigma_2(z),\ \tau_{12}(z)$ are the stresses, then

$$N_1=\int_{-h/2}^{h/2}\sigma_1(z)\,dz,\qquad N_2=\int_{-h/2}^{h/2}\sigma_2(z)\,dz,\qquad T=\int_{-h/2}^{h/2}\tau_{12}(z)\,dz,$$

$$M_1=\int_{-h/2}^{h/2}\sigma_1(z)\,z\,dz,\qquad M_2=\int_{-h/2}^{h/2}\sigma_2(z)\,z\,dz,\qquad H=\int_{-h/2}^{h/2}\tau_{12}(z)\,dz. \qquad (1,2)$$

For a linear homogeneous and isotropic Maxwell medium in the plane stress state we have

$$\dot{\varepsilon}_1(z) = \frac{D_0}{2\mu}\sigma_1(z) + \frac{1}{3}\left(\frac{D_1}{3K_\nu} - \frac{D_0}{2\mu}\right)(\sigma_1(z) + \sigma_2(z)) =$$

$$= \frac{1}{3}\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\sigma_1(z) - \frac{1}{3}\left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right)\sigma_2(z),$$

$$\dot{\varepsilon}_2(z) = \frac{D_0}{2\mu}\sigma_2(z) + \frac{1}{3}\left(\frac{D_1}{3K_\nu} - \frac{D_0}{2\mu}\right)(\sigma_1(z) + \sigma_2(z)) =$$

$$= \frac{1}{3}\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\sigma_2(z) - \frac{1}{3}\left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right)\sigma_1(z), \qquad \dot{\gamma}_{12}(z) = \frac{D_0}{\mu}\tau_{12}(z), \tag{1,3}$$

where $\varepsilon_1(z), \varepsilon_2(z)$ are the relative strains of the surface $z = \text{const}$ along the axes x and y; $\gamma_{12}(z)$ is the shear angle of the surface $z = \text{const}$; $K_\nu = \frac{2}{3}\mu + \lambda$ is the bulk viscosity; μ, λ are rigidity coefficients; G is the shear modulus; $K_e = 2G(1+\nu)/3(1-2\nu)$; ν is Poisson' s ratio; the dot denotes

differentiation with respect to time t. Here the time operators are

$$D_0 = 1 + \frac{\mu}{G}\frac{\partial}{\partial t}, \qquad D_1 = 1 + \frac{K_\nu}{K_e}\frac{\partial}{\partial t}. \tag{1,4}$$

According to (1,3), the stresses are equal to

$$\left(\frac{D_0}{2\mu} + \frac{2}{3}\frac{D_1}{K_\nu}\right) D_0\sigma_1(z) = 2\mu\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\dot{\varepsilon}_1(z) + \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right)\dot{\varepsilon}_2(z)\right],$$

$$\left(\frac{D_0}{2\mu} + \frac{2}{3}\frac{D_1}{K_\nu}\right) D_0\sigma_2(z) = 2\mu\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\dot{\varepsilon}_2(z) + \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right)\dot{\varepsilon}_1(z)\right],$$

$$D_0\tau_{12}(z) = \dot{\gamma}_{12}(z). \tag{1,5}$$

The Kirchhoff–Love hypotheses lead to the following expressions for the strains:

$$\varepsilon_1(z) = \varepsilon_1 - z\chi_1, \qquad \varepsilon_2(z) = \varepsilon_2 - z\chi_2, \qquad \gamma_{12}(z) = \gamma_{12} - 2z\chi_{12}, \tag{1,6}$$

where the relative strains ε_1 and ε_2 and the shear angle γ_{12} of the middle surface for a shallow shell with small displacements have the form

$$\varepsilon_1 = \frac{\partial u}{\partial x} - \frac{w}{R_1}, \qquad \varepsilon_2 = \frac{\partial v}{\partial y} - \frac{w}{R_2}, \qquad \gamma_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{1,7}$$

$$\chi_1 = \frac{\partial^2 w}{\partial x^2}, \qquad \chi_2 = \frac{\partial^2 w}{\partial y^2}, \qquad \chi_{12} = \frac{\partial^2 w}{\partial x\,\partial y}.$$

We substitute (1,5) into (1,2). Then, taking (1,6) into account, we obtain

$$\left(\frac{D_0}{2\mu} + \frac{2}{3}\frac{D_1}{K_\nu}\right) D_0 M_1 = -\frac{\mu h^3}{6}\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\dot{\chi}_1 + \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right)\dot{\chi}_2\right],$$

$$\left(\frac{D_0}{2\mu} + \frac{2}{3}\frac{D_1}{K_\nu}\right) D_0 M_2 = -\frac{\mu h^3}{6}\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\dot{\chi}_2 + \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right)\dot{\chi}_1\right],$$

$$D_0 H = -\frac{\mu h^3}{6}\dot{\chi}_{12}, \qquad D_0 T = \mu h \dot{\gamma}_{12}, \tag{1,8}$$

$$\left(\frac{D_0}{2\mu} + \frac{2}{3}\frac{D_1}{K_\nu}\right) D_0 N_1 = 2\mu h\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\dot{\varepsilon}_1 + \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right)\dot{\varepsilon}_2\right],$$

$$\left(\frac{D_0}{2\mu} + \frac{2}{3}\frac{D_1}{K_\nu}\right) D_0 N_2 = 2\mu h\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\dot{\varepsilon}_2 + \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right)\dot{\varepsilon}_1\right].$$

We substitute (1,6) into (1,5), multiply both sides of the expression by dz, and then by $z\,dz$, and integrate the result over the thickness of the shell. We find

$$\dot{\varepsilon}_1 = \frac{1}{3h}\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right) N_1 - \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right) N_2\right],$$

$$\dot{\varepsilon}_2 = \frac{1}{3h}\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right) N_2 - \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right) N_1\right],$$

$$\dot{\gamma}_{12} = \frac{1}{\mu h} D_0 T, \qquad \dot{\chi}_{12} = -\frac{6}{\mu h^3} D_0 H, \tag{1,9}$$

$$\dot{\chi}_1 = -\frac{4}{h^3}\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right) M_1 - \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right) M_2\right],$$

$$\dot{\chi}_2 = -\frac{4}{h^3}\left[\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right) M_2 - \left(\frac{D_0}{2\mu} - \frac{D_1}{3K_\nu}\right) M_1\right].$$

From the first three relations (1,7) we obtain the strain-compatibility equation

$$\frac{\partial^2 \varepsilon_1}{\partial y^2} + \frac{\partial^2 \varepsilon_2}{\partial x^2} - \frac{\partial^2 \gamma_{12}}{\partial x\,\partial y} = -\frac{1}{R_1}\frac{\partial^2 w}{\partial y^2} - \frac{1}{R_2}\frac{\partial^2 w}{\partial x^2}, \tag{1,10}$$

and after differentiation with respect to time, the compatibility equation for the strain rates,

$$\frac{\partial^2 \dot{\varepsilon}_1}{\partial y^2} + \frac{\partial^2 \dot{\varepsilon}_2}{\partial x^2} - \frac{\partial^2 \dot{\gamma}_{12}}{\partial x\,\partial y} = -\frac{1}{R_1}\frac{\partial^2 \dot{w}}{\partial y^2} - \frac{1}{R_2}\frac{\partial^2 \dot{w}}{\partial x^2}. \tag{1,11}$$

The first two equilibrium equations (1,1), for $X = Y = 0$, are satisfied by the stress function F, defined as

$$N_1 = \frac{\partial^2 F}{\partial y^2}, \qquad N_2 = \frac{\partial^2 F}{\partial x^2}, \qquad T = -\frac{\partial^2 F}{\partial x\,\partial y}. \tag{1,12}$$

Substitute the fourth and fifth equations of (1.1) into the third, and substitute there, for M_1, M_2, H, their values according to relations (1.8), and for N_1 and N_2, their expressions through the force function (1.12). We insert the first three dependences (1.9) into (1.11), taking (1.12) into account. As a result we obtain

$$\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\nabla^4 w - \frac{6}{h^3}\left(\frac{D_0}{2\mu} + \frac{2}{3}\frac{D_1}{K_\nu}\right)\frac{D_0}{\mu}\left[\frac{1}{R_1}\frac{\partial^2 F}{\partial y^2} + \frac{1}{R_2}\frac{\partial^2 F}{\partial x^2} + Z\right] = 0, \tag{1.13}$$

$$\left(\frac{D_0}{\mu} + \frac{D_1}{3K_\nu}\right)\nabla^4 F = -3h\left(\frac{1}{R_1}\frac{\partial^2 w}{\partial y^2} + \frac{1}{R_2}\frac{\partial^2 w}{\partial x^2}\right) \tag{1.14}$$

$$\left(\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right).$$

By Z in (1.13) one should understand the reduced transverse load, which in the case of a shell with initial forces N_1^0, N_2^0, T^0 and the action of the external medium is equal to

$$Z = N_1^0\frac{\partial^2 w}{\partial x^2} + N_2^0\frac{\partial^2 w}{\partial y^2} + 2T^0\frac{\partial^2 w}{\partial x \partial y} + \frac{\gamma h}{g}\ddot{w} + p_c, \tag{1.15}$$

where γ is the specific weight of the shell material; g is the acceleration of gravity; p_c is the pressure of the medium on the shell surface.

From (1.13)–(1.14) one can obtain one differential equation with respect to the deflection

$$\left(\frac{D_0}{\mu}+\frac{D_1}{3K_\nu}\right)^2\nabla^8 w+\frac{6}{h^3}\left(\frac{D_0}{2\mu}+\frac{2}{3}\frac{D_1}{K_\nu}\right)\frac{D_0}{\mu}\left[3h\left(\frac{1}{R_1^2}\frac{\partial^4 w}{\partial x^4}+\frac{2}{R_1R_2}\frac{\partial^4 w}{\partial x^2\partial y^2}\right.\right.$$

$$\left.\left.+\frac{1}{R_2^2}\frac{\partial^4 w}{\partial x^4}\right)-\left(\frac{D_0}{\mu}+\frac{D_1}{3K_\nu}\right)\nabla^4\left(N_1^0\frac{\partial^2 w}{\partial x^2}+N_2^0\frac{\partial^2 w}{\partial y^2}+2T^0\frac{\partial^2 w}{\partial x\partial y}+\frac{\gamma h}{g}\ddot{w}+p_c\right)\right]=0. \tag{1.16}$$

If a new function Φ is introduced by means of

$$F=3h\left(\frac{1}{R_1}\frac{\partial^2\Phi}{\partial y^2}+\frac{1}{R_2}\frac{\partial^2\Phi}{\partial x^2}\right),\qquad w=-\left(\frac{D_0}{\mu}+\frac{D_1}{3K_\nu}\right)\nabla^4\Phi, \tag{1.17}$$

then equation (1.14) is identically satisfied, and (1.13) takes the form

$$\left(\frac{D_0}{\mu}+\frac{D_1}{3K_\nu}\right)^2\nabla^8\Phi+\frac{18}{h^2}\left(\frac{D_0}{2\mu}+\frac{2}{3}\frac{D_1}{K_\nu}\right)\frac{D_0}{\mu}\left\{\frac{1}{R_1^2}\frac{\partial^4\Phi}{\partial y^4}+\frac{2}{R_1R_2}\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{1}{R_2^2}\frac{\partial^4\Phi}{\partial x^4}\right.$$

$$\left.+\frac{1}{3h}\left[N_1^0\frac{\partial^2 w}{\partial x^2}+N_2^0\frac{\partial^2 w}{\partial y^2}+2T^0\frac{\partial^2 w}{\partial x\partial y}-\frac{\gamma h}{g}\left(\frac{D_0}{\mu}+\frac{D_1}{3K_\nu}\right)\nabla^4\Phi+p_c\right]\right\}=0.$$

2. Oscillations of a cylindrical shell in a supersonic gas flow

Let a cylindrical shell be subjected to internal transverse pressure p and be externally washed by a supersonic gas flow. Then

$$R_1=\infty,\qquad R_2=R,\qquad y=R\varphi,\qquad T^0=0,\qquad N_1^0=\frac{1}{2}pR,\qquad N_2^0=pR, \tag{2.1}$$

$$-q=q_1\frac{\partial w}{\partial t}+q_2\frac{\partial w}{\partial x}\qquad\left(q_1=\frac{\rho U}{\sqrt{M^2-1}},\qquad q_2=\frac{\rho U^2}{\sqrt{M^2-1}}\right).$$

Here q is the additional pressure in the gas flow due to the deviation of the shell from the undisturbed cylindrical form during oscillations, corresponding to the theory of a stationary supersonic flow; U is the velocity of the undisturbed flow; $M = U/c$; c is the speed of propagation of sound in the undisturbed flow; ρ is the density of the flow.

We substitute (2.1) into equations (1.13)–(1.14) and seek solutions of these equations in the form of traveling waves

$$w = w_0 e^{i(\omega t - kx)} \cos n\varphi, \qquad F = F_0 e^{i(\omega t - kx)} \cos n\varphi, \tag{2.2}$$

where w_0, F, k are constants; n is the number of half-waves in the circumferential direction; ω is the circular frequency of oscillation of the shell in the flow.

Substitution of (2.2) into (1.13)–(1.14) leads to the following equation for the reduced frequency ω^* of oscillations of the shell in a flow:

$$\omega^{*5} - i\omega^{*4} a_4 - \omega^{*3}(a_{32} - ia_{31}) + \omega^{*2}(a_{22} + ia_{21}) + \omega^*(a_{12} - ia_{11}) + + a_{02} + ia_{01} = 0. \tag{2.3}$$

Here

$$a_4 = \frac{1}{3(1-\nu^2)\pi}\left(\frac{3-\nu-\nu^2}{\mu^*} + \frac{3+2\nu-\nu^2}{3K_\nu^*}\right) + \frac{q_1^*}{\pi}, \qquad a_{31} = \frac{q_2^* hk}{\pi^2},$$

$$a_{32} = \frac{\theta}{\pi^2}\left[\frac{h^2}{R^2}\frac{k^4}{(k^2+n^2/R^2)^4} + \frac{h^4}{12(1-\nu^2)}\left(k^2 + \frac{n^2}{R^2}\right)^2\right] + N_1^*\frac{k^2R^2}{\pi^2} + N_2^*\frac{n^2}{\pi^2} + \\ + \frac{1}{18(1-\nu^2)\pi^2}\left(\frac{11-4\nu}{2\mu^{*2}} + 2\frac{7+\nu}{3K_\nu^*\mu^*} + \frac{4}{9}\frac{1+\nu}{K_\nu^{*2}}\right) + \\ + \frac{1}{3(1-\nu^2)}\frac{q_1^*}{\pi^2}\left(\frac{3-\nu-\nu^2}{\mu^*} + \frac{3+2\nu-\nu^2}{3K_\nu^*}\right),$$

$$a_{22} = \frac{q_2^* hk}{3(1-\nu^2)\pi^3}\left(\frac{3-\nu-\nu^2}{\mu^*} + \frac{3+2\nu-\nu^2}{3K_\nu^*}\right),$$

$$a_{21} = \frac{1}{3(1-\nu^2)\pi^3}\left\{\theta\left[\frac{h^2}{R^2}\frac{k^4}{(k^2+n^2/R^2)^2}\left(\frac{2-\nu}{\mu^*} + 2\frac{1+\nu}{3K_\nu^*}\right) + \right.\right. \\ \left. + \frac{1}{6}h^4\left(k^2 + \frac{n^2}{R^2}\right)^2\left(\frac{1}{\mu^*} + \frac{1}{3K_\nu^*}\right)\right] + \left(\frac{3-\nu-\nu^2}{\mu^*} + \frac{3+2\nu-\nu^2}{3K_\nu^*}\right)(N_1^* R^2 k^2 + \\ \left. + N_2^* n^2) + \frac{1}{6}\frac{1}{\mu^*}\left(\frac{1}{2\mu^*} + \frac{2}{3K_\nu^*}\right)\left(\frac{1}{\mu^*} + \frac{1}{3K_\nu^*}\right) + \frac{1}{6}q_1^*\left(\frac{11-4\nu}{2\mu^{*2}} + 2\frac{7+\nu}{3K_\nu^*\mu^*} + \frac{4}{9}\frac{1+\nu}{K_\nu^{*2}}\right)\right\},$$

$$a_{12} = \frac{1}{6(1-\nu^2)\pi^4}\left\{\theta\left[\frac{h^2}{R^2}\frac{k^4}{(k^2+n^2/R^2)^2}\frac{1}{\mu^*}\left(\frac{1}{2\mu^*}+\frac{2}{3K_\nu^*}\right)+\right.\right.$$
$$\left.+\frac{1}{48(1-\nu^2)}h^4\left(k^2+\frac{n^2}{R^2}\right)^2\left(\frac{1}{\mu^*}+\frac{1}{3K_\nu^*}\right)^2\right]+$$
$$+\frac{1}{3}\left(\frac{11-7\nu}{2\mu^{*2}}+2\frac{7+\nu}{3K_\nu^*\mu^*}+\frac{4}{9}\frac{1+\nu}{K_\nu^{*2}}\right)(N_1^*R^2k^2+N_2^*n^2)+$$
$$\left.+\frac{1}{3}\frac{q_1^*}{\mu^*}\left(\frac{1}{2\mu^*}+\frac{2}{3K_\nu^*}\right)\left(\frac{1}{\mu^*}+\frac{1}{3K_\nu^*}\right)\right\},$$

$$a_{11} = \frac{q_2^*hk}{18(1-\nu^2)\pi^4}\left(\frac{11-4\nu}{2\mu^{*2}}+2\frac{7+\nu}{3K_\nu^*\mu^*}+\frac{4}{9}\frac{1+\nu}{K_\nu^{*2}}\right),$$

$$a_{02} = \frac{q_2^*hk}{18(1-\nu^2)\pi^5}\frac{1}{\mu^*}\left(\frac{1}{2\mu^*}+\frac{2}{3K_\nu^*}\right)\left(\frac{1}{\mu^*}+\frac{1}{3K_\nu^*}\right),$$

$$a_{01} = \frac{1}{18(1-\nu^2)\pi^5}\frac{1}{\mu^*}\left(\frac{1}{2\mu^*}+\frac{2}{3K_\nu^*}\right)\left(\frac{1}{\mu^*}+\frac{1}{3K_\nu^*}\right)(N_1^*R^2k^2+N_2^*n^2).$$

$$\omega^* = \frac{\omega h}{\pi c},\qquad \theta = \frac{gE}{\gamma c^2},\qquad \mu^* = \frac{\mu c}{Eh},\qquad K_\nu^* = \frac{K_\nu c}{Eh},\qquad q_1^* = \frac{q_1 g}{\gamma c} = \frac{\rho g}{\gamma}\frac{M}{\sqrt{M^2-1}},$$

$$q_2^* = \frac{q_2 g}{\gamma c^2} = \frac{\rho g}{\gamma}\frac{M^2}{\sqrt{M^2-1}},\qquad N_1^* = \frac{gN_1^0 h}{\gamma c^2R^2},\qquad N_2^* = \frac{gN_2^0 h}{\gamma c^2 R^2}.$$

In (2.3) the reduced frequency is a complex quantity.

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Note: Figure translations are in progress. See original paper for figures.