Mathematics

Corresponding Member of the Academy of Sciences of the USSR
Yu. V. Linnik

1961

SovietRxiv

Abstract

Full Text

Mathematics

Corresponding Member of the Academy of Sciences of the USSR Yu. V. Linnik

New Variants and Applications of the Dispersion Method in Binary Additive Problems

The dispersion method in binary additive problems, constructed and developed by the author in the works ([1–6]), had as its aim the construction of asymptotic formulas for solving binary additive equations of the form

$$n = \varphi + D'\nu \qquad (n > 0) \tag{1}$$

or

$$a = \varphi - D'\nu, \qquad a > 0 \quad \text{or} \quad a < 0, \tag{2}$$

where $\{\varphi\}$ forms (with possible repetitions) a certain sequence of natural numbers; D' runs through a certain "sufficiently dense" system of natural numbers; ν is a sequence of numbers which may be very sparse, but sufficiently well distributed in intervals of arithmetic progressions with slowly increasing differences. Problems of the form (1) may be called definite; they sometimes arise in the absence of conditions on the summands, which automatically must be bounded, whereas problems of the form (2) we shall call indefinite; they require restrictions on the summands.

Typical examples of equations (1) and (2) are:

$$n = Q(x, y) + p_1 p_2^b, \quad n = Q(x, y) + p_1 \cdot 2^b,$$

where $Q(x, y)$ is a quadratic form; p_1, p_2 are primes; b is a constant.

With the aid of one or another variant of the sieve of Eratosthenes we can reduce to (1) or (2) the solution of the equations

$$n = \varphi + p, \qquad a = \varphi - p, \tag{3}$$

where p denotes prime numbers. Typical examples will be:

$$n = x^2 + y^2 + p$$

(the Hardy-Littlewood equation);

$$-1 = xy - p \tag{4}$$

(E. C. Titchmarsh' s problem on the asymptotic behavior of the number of divisors of shifted prime numbers);

$$1 = xy - x_1 x_2 \dots x_k, \tag{5}$$

where $x_i \geqslant 1$ (the additive divisor problem).

The basic scheme of the dispersion method we shall call the scheme set forth in ([4]). Let us dwell on equation (1). First of all, from the system

from the numbers D' we select the numbers having too many repetitions. The number of solutions of (1) corresponding to such D' should be relatively small, and we must be able to discard it. Let the remaining D' have no more than M repetitions. We divide the ranges of variation of the numbers D' and v into pairs of narrow zones: (D), (v), so that, with an admissible error in the number of solutions of (1), the numbers D' and v may be regarded as varying independently in these zones.

Let

$$U(m) = \sum_{\varphi=m} 1.$$

The number of solutions of (1) for D' and v varying in their zones will be expressed by the formula

$$\sum_{D'\in(D)} \sum_{v\in(v)} U(n - D'v). \tag{6}$$

Let D be any integer, not necessarily from the system of numbers D'. For a given D consider the equation

$$n = \varphi + Dv. \tag{7}$$

The next step is to find, by some heuristic means or other, the expected number of solutions of (7),

$$A(n, D). \tag{8}$$

After this one forms the zonal variance of the number of solutions of (1) under the presumed asymptotic $A(n, D)$. It has the form

$$V' = \sum_{D'\in(D)} \left(\sum_{v\in(v)} U(n - D'v) - A(n, D') \right)^2. \tag{9}$$

The main subsequent task is to estimate the variance (9) from above, having in view the further application of the obvious analogue of Chebyshev' s inequality to the solution of equation (1). Since D' has no more than M repetitions, we have

$$V' \ll M \sum_{D'' \in (D)} \left(\sum_{v \in (v)} U(n - D''v) - A(n, D'') \right)^2, \tag{10}$$

where D runs through the same values as D', but without repetitions. Following the main idea of I. M. Vinogradov' s method as applied to the estimation of double sums, we have

$$V' \ll MV, \qquad \text{where} \qquad V = \sum_{D \in (D)} \left(\sum_{v \in (v)} U(n - Dv) - A(n, D) \right)^2,$$

D runs through all the numbers of the zone (D) consecutively (sometimes it is more advantageous to make D run through an arithmetic progression containing the value D''). The number V will be called the complete zonal variance. The estimation of V reduces (see (4)) to the asymptotic calculation of the number of solutions of the "main equation"

$$v_1(n - \varphi_1) = v_2(n - \varphi_2); \qquad v_1 \neq v_2; \qquad \varphi_i \in \{\varphi\} \tag{11}$$

under the conditions

$$n - \varphi_1 \equiv 0 \pmod{v_2}; \qquad \frac{n - \varphi_2}{v_2} \in (D). \tag{12}$$

We shall set forth some variants of the basic scheme and their application. One can generalize the main equation by considering the equation

$$n = \varphi + D'v + \beta(v), \tag{13}$$

where $\beta(v)$ is a given integer-valued function of v. Example:

$$n = \varphi(x, y) + p_1 p_2^a + p_2^b,$$

where $\varphi(x, y)$ is a quadratic form, p_i are primes, and a, b are constants. Here, as the complete zonal variance one should take

$$V = \sum_{D \in (D)} \left(U(n - D\nu - \beta(\nu)) - A(n, D) \right)^2. \tag{14}$$

Then the basic equation will take the form

$$\nu_1 \varphi_1 - \nu_2 \varphi_2 = n(\nu_1 - \nu_2) + \beta(\nu_1)\nu_2 - \beta(\nu_2)\nu_1; \qquad \nu_1 \neq \nu_2 \tag{15}$$

under the additional conditions

$$n - \varphi_1 - \beta(\nu_2) \equiv 0 \pmod{\nu_2}; \qquad \frac{n - \varphi_1 - \beta(\nu_2)}{\nu_2} \in (D). \tag{16}$$

It is advisable to solve equation (15) with fixed ν_1, ν_2, then collecting the solutions over pairs ν_1, ν_2 $(\nu_1 \neq \nu_2)$.

In computing the complete zonal variance, the greatest difficulty is presented by deriving the asymptotic formula for the solutions of the basic equation (11) (or (15)); however, the calculations connected with finding $A(n, D)$ and processing the terms containing it are also rather laborious and cumbersome. In view of this, the concept of the variance of the difference and of the covariance of solutions proves very useful. Suppose two equations are given:

$$n_1 = \varphi + D'\nu' \tag{17}$$

and an equation independent of it:

$$n_2 = \psi + D''\nu'', \tag{18}$$

where D', D'' independently run through one and the same system of numbers; the same applies to ν', ν''; $\{\varphi\}$, $\{\psi\}$ independently run through one and the same sequences of natural numbers. For simplicity, suppose that D', D'' have no repetitions. The zonal variance of the difference of the solutions of (17) and (18) will be called

$$V' = \sum_{D' \in (D)} \left(\sum_{\nu \in (\nu)} U_1(n - D'\nu) - \sum_{\nu \in (\nu)} U_2(n_2 - D'\nu) \right)^2, \tag{19}$$

where

$$U_1(m) = \sum_{\varphi = m} 1; \qquad U_2(m) = \sum_{\psi = m} 1.$$

We have $V' \ll V$, where V is the complete zonal variance of the difference

$$V = \sum_{D \in (D)} \left(\sum_{\nu \in (\nu)} U_1(n_1 - D\nu) - \sum_{\nu \in (\nu)} U_2(n_2 - D\nu) \right)^2. \tag{20}$$

Let us note that here the quantity $A(n, D)$ is absent; finding and processing it require cumbersome computations. The principal role in calculating (20) is played by the covariance of the number of solutions

$$\operatorname{Cov}(n_1, n_2) = \sum_{D \in (D)} \sum_{\nu \in (\nu)} U_1(n_1 - D\nu) \sum_{\nu \in (\nu)} U_2(n_2 - D\nu). \tag{21}$$

For calculating the covariance, the principal role is played by the equation

$$\nu_1 \varphi_1 - \nu_2 \varphi_2 = \nu_1 n_1 - \nu_2 n_2 \tag{22}$$

under the conditions

$$n - \varphi_1 \equiv 0 \pmod{\nu_2}; \qquad \frac{n - \varphi_1}{\nu_2} \in D.$$

It is often possible, for a given number n_1, to isolate such simply describable numbers n_2 for which the variance of the difference of the numbers of solutions is relatively small. An analogue of Chebyshev' s inequality then shows that equations (17)

and (18), with an admissible relative error, have the same number of solutions. In this case we shall call the number n_2 coherent with the number n_1. If n' runs through the numbers coherent with n_1, then, instead of the binary problem (17), it suffices to solve the ternary additive problem

$$n' - \varphi - D'v = 0. \tag{23}$$

The advantage of this approach is that it is not necessary to compute $A(n, D)$ and to process the terms containing it.

Let us give some applications of the new variant of the dispersion method. For the additive divisor problem, treated in [3], we obtain the formula

$$\sum_{m \le n} \tau(m+1)\tau_n(m) = k! A_n S_k n (\ln n)^k + Bn(\ln n)^{k-1} (\ln \ln n)^{k^4}, \tag{24}$$

where A_n, S_k are the constants explained in [3].

For E. C. Titchmarsh' s divisor problem for shifted prime numbers we obtain the formula

$$\sum_{p \le n} \tau(p-1) = \frac{315\zeta(3)}{2\pi^4} n + R(n), \tag{25}$$

where $R(n) = O\left(\frac{n}{(\ln n)^{\alpha}}\right)$ and $\alpha > 0$ is any constant less than 1.

Received
1 II 1961

References

[1] Yu. V. Linnik, DAN, **120**, No. 5, 960 (1958).
[2] Yu. V. Linnik, DAN, **123**, No. 6, 975 (1958).
[3] Yu. V. Linnik, Proc. of the Intern. Congress of Math., Edinburgh, 1958, Cambridge, 1960, p. 313.
[4] Yu. V. Linnik, Matem. sborn., **51** (93), No. 2, 129 (1960).
[5] Yu. V. Linnik, Acta Math. Acad. Sci. Hungaricae, **4**, No. 3–4, 225 (1953).
[6] Yu. V. Linnik, Izv. AN SSSR, ser. matem., **24**, No. 5, 629 (1960).

Note: Figure translations are in progress. See original paper for figures.