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Abstract

Full Text

Mathematics

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On Periodic Solutions of Boundary-Value Problems Without Initial Conditions for Quasilinear Parabolic Equations

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Time-periodic solutions of boundary-value problems for nonlinear one-dimensional parabolic equations with linear principal part were studied in a number of papers by D. Kh. Karimov ⁽¹⁾ and G. Prodi ^(2,3).

In the present note, under certain conditions, by the method of finite differences in the form proposed by E. Rothe ⁽⁴⁾, theorems are proved on the existence of solutions periodic in t of certain boundary-value problems for the equation

$$u_t = a(x, t, u)u_{xx} + f(x, t, u, u_x), \quad (1)$$

as well as a theorem on the existence of a solution periodic in t of the first boundary-value problem for the equation

$$u_t = \sum_{i,j=1}^m a_{ij}(x, t, u)u_{x_i x_j} + \sum_{i=1}^m a_i(x, t, u)u_{x_i} + a(x, t, u). \quad (2)$$

Let us formulate the results obtained:

Theorem 1. In the strip

$$Q = \{0 \leq x \leq l, -\infty < t < +\infty\}$$

there exists a continuous solution, periodic in t with period T , of the boundary-value problem

$$u_t = a(x, t, u)u_{xx} + f(x, t, u, u_x), \quad u(0, t) = u(l, t) = 0, \quad (I)$$

possessing inside Q the continuous derivatives entering into equation (1), if the following conditions are satisfied:

1. In the layer

$$S_1 = \{0 \leq x \leq l, -\infty < t < +\infty, -\infty < u < +\infty\}$$

the functions $a(x, t, u)$ and $f(x, t, u, 0)$ satisfy the inequalities

$$a(x, t, u) \geq \alpha \quad (\alpha = \text{const} > 0),$$

$$\int_0^1 \frac{\partial f(x, t, \tau u, 0)}{\partial u} d\tau \leq -c_0 \quad (c_0 = \text{const} > 0).$$

2. The functions $a(x, t, u)$ and $f(x, t, u, p)$ in the domain

$$S_2 = \{0 \leq x \leq l, -\infty < t < +\infty, -C_0 \leq u \leq C_0, -\infty < p < +\infty\}$$

are continuous and possess continuous derivatives of the form

$$\partial^\nu a / \partial x^{\nu_1} \partial u^{\nu_2}, \quad \partial^\mu f / \partial x^{\mu_1} \partial u^{\mu_2} \partial p^{\mu_3}$$

$$(\nu = 1, \dots, 4; \mu = 1, \dots, 4).$$

3. In S_2 the function $f(x, t, u, p)$ and its first derivatives with respect to x and u have, with respect to p , growth order less than two, while the function $\partial f / \partial p$ has, with respect to p , growth order less than one.

4. The functions $a(x, t, u)$ and $f(x, t, u, p)$ are periodic in t with period T .

Theorem 2. In the strip Q there exists a solution, continuous together with its first derivative with respect to x and periodic in t with period T , of the boundary-value problem

$$u_t = a(x, t, u)u_{xx} + f(x, t, u, u_x),$$

$$u_x(0, t) = \varphi_1(t, u(0, t)), \tag{II}$$

$$u_x(l, t) = \varphi_2(t, u(l, t)).$$

possessing inside Q continuous derivatives entering into equation (1), if the following conditions are fulfilled:

1. Conditions 1, 3, 4 of Theorem 1.

2. The functions $a(x, t, u)$ and $f(x, t, u, p)$ in S_2 are continuous and possess continuous derivatives of second order with respect to x, u, p , having bounded derivatives with respect to t, u, p of first order.

3. The functions $\varphi_1(t, u)$ and $\varphi_2(t, u)$ in the domain $S_3 = \{-\infty < t < +\infty, -C_0 \leq u \leq C_0\}$ have continuous derivatives of second order with respect to u and of first order with respect to t , and satisfy in S_3 the conditions

$$\frac{\partial \varphi_1}{\partial u} > 0, \quad \frac{\partial \varphi_2}{\partial u} < 0, \quad \varphi_1(t, 0) = \varphi_2(t, 0) = 0.$$

4. The functions $\varphi_1(t, u)$ and $\varphi_2(t, u)$ are periodic in t with period T .

Theorem 3. In the strip Q there exists a solution of the boundary-value problem

$$u_t = a(x, t, u)u_{xx} + f(x, t, u, u_x),$$

$$u_t(0, t) = \varphi_1(t, u(0, t), u_x(0, t)),$$

$$u_t(l, t) = \varphi_2(t, u(l, t), (u_x(l, t))), \quad (\text{III})$$

continuous together with its first derivatives with respect to t and x and periodic in t with period T , possessing inside Q a continuous second derivative with respect to x , if the following conditions are fulfilled:

1. Conditions 1, 3 and 4 of Theorem 1.
2. The functions $a(x, t, u)$ and $f(x, t, u, p)$ in S_2 possess continuous derivatives of third order with respect to x, u, p ; both the functions $a(x, t, u)$ and $f(x, t, u, p)$, and their derivatives up to and including second order with respect to x, u , and p , have bounded derivatives of first order with respect to t .
3. The functions $\varphi_1(t, u, p)$ and $\varphi_2(t, u, p)$ in the domain

$$S_4 = \{-\infty < t < +\infty, -C_0 \leq u \leq C_0, -\infty < p < +\infty\}$$

have continuous derivatives of second order with respect to u and p , and of first order with respect to t .

4. The functions $\varphi_1(t, u, p)$ and $\varphi_2(t, u, p)$ have order of growth in p less than two and satisfy the conditions

$$\frac{\partial \varphi_1}{\partial p} \geq \beta_1 > 0, \quad \frac{\partial \varphi_2}{\partial p} \leq -\beta_2 < 0, \quad \frac{\partial \varphi_1}{\partial u} \leq \gamma_1 < 0, \quad \frac{\partial \varphi_2}{\partial u} \leq \gamma_2 < 0,$$

where $\beta_1, \beta_2, \gamma_1$ and γ_2 are constants.

5. The functions $\varphi_1(t, u, p)$ and $\varphi_2(t, u, p)$ are periodic in t with period T .

Since the proofs of Theorems 1-3 have many features in common, we shall confine ourselves to a description of the scheme of proof of Theorem 1.

Let t_0 be any number from $(-\infty, \infty)$, and let

$$Q_T = \{0 \leq x \leq l, t_0 \leq t \leq T + t_0\}$$

be a rectangle of height T . If we prove the existence in Q_T of a function $u(x, t)$ having the properties indicated in Theorem 1 and satisfying the condition

$$u(x, t_0 + T) = u(x, t_0),$$

then Theorem 1 will be proved.

The problem just formulated will be called problem (I').

Choosing a sufficiently small step h , draw in Q_T a series of straight lines $t = kh$ ($k = 1, \dots, N$), and put problem (I') in correspondence with the problem of determining in $[0, l]$ the vector

$$\mathbf{u} = (u_1, \dots, u_N)$$

as a twice continuously differentiable solution of the boundary-value problem

$$a_k u_{kxx} + b_k u_{kx} + c_k u_k + f_k = \frac{u_k - u_{k-1}}{h}; \quad (3)$$

$$u_k(0) = u_k(l) = 0 \quad (k = 1, \dots, N). \quad (4)$$

It should be noted that for $k = 1$ the right-hand side of (3) has the form $\frac{u_1 - u_N}{h}$. We have used the following abbreviated notation:

$$a_k \equiv a_k(x, u_k) = a(x, t_k, u_k),$$

$$b_k \equiv b_k(x, u_k, u_{kx}) = \int_0^1 \frac{\partial f(x, t_k, u_k, \tau v)}{\partial v} \Big|_{v=u_{kx}} d\tau,$$

$$c_k \equiv c_k(x, u_k) = \int_0^1 \frac{\partial f(x, t_k, \tau v, 0)}{\partial v} \Big|_{v=u_k} d\tau,$$

$$f_k \equiv f_k(x) = f(x, t_k, 0, 0), \quad t_k = kh \quad (k = 1, \dots, N).$$

The existence of a solution of class $C^{(2)}[0, l]$ of the boundary value problem (3), (4) is proved using the Leray-Schauder theorem ⁽⁵⁾ on the existence of a fixed point of an operator equation with a parameter

$$\mathbf{u} = A(\mathbf{u}, \lambda).$$

For problem (3), (4), $A(\mathbf{v}, \lambda)$ is defined as the operator which assigns to each vector $\mathbf{v} \in C^{(2)}[0, l]$ the vector $\mathbf{u} \in C^{(2)}[0, l]$ as the solution of the boundary value problem

$$a_k(x, \lambda v_k) u_{kxx} + b_k(x, \lambda v_k, \lambda v_{kx}) u_{kx} + c_k(x, \lambda v_k) u_k + f_k(x) = \frac{u_k - u_{k-1}}{h},$$

$$u_k(0) = u_k(l) = 0 \quad (k = 1, \dots, N).$$

The existence of $A(\mathbf{v}, \lambda)$ is established by the method of integral equations. After proving the solvability of the boundary value problem (3), (4) for arbitrarily small $|h|$, estimates uniform with respect to h are established:

$$|u_{kx^\nu}| \leq C_\nu \quad (\nu = 0, \dots, 4), \tag{5}$$

where the estimates with $\nu = 0, 1, 2$ are valid on $[0, l]$, and the estimates with $\nu = 3, 4$ are valid inside $[0, l]$. The estimates (5), by means of simple arguments, make it possible to complete the proof of Theorem 1.

We particularly note the following circumstance. Both the estimates needed to verify the conditions of the Leray-Schauder theorem and the estimates needed to prove the convergence of the solutions of the differential-difference problems to the solutions of the corresponding boundary value problems (I), (II), (III) are obtained by the method of S. N. Bernstein ⁽⁶⁾, analogously to how this is done in ⁽⁷⁻¹⁰⁾.

Passing to the multidimensional equation (2), denote by $Q = D \times (-\infty, \infty)$ the straight cylinder with lateral surface S . We shall assume that the bounded m -dimensional domain \bar{D} belongs to the class $A^{(2,\lambda)}$ ⁽¹¹⁾.

Theorem 4. Let, for $(x, t) \in \bar{Q}$ and $u \in (-\infty, \infty)$, the inequalities

$$-\frac{\partial a(x, t, u)}{\partial u} \geq c_0 \quad (c_0 = \text{const} > 0); \tag{6}$$

$$\sum_{i,j=1}^m a_{ij}(x, t, u) \xi_i \xi_j \geq \alpha \sum_{i=1}^m \xi_i^2 \quad (\alpha = \text{const} > 0), \tag{7}$$

hold, and for $(x, t) \in \bar{Q}$ and $|u| \leq C_0$, where $C_0 > 0$ is some constant, the inequality

$$\max_{(x,t,u)} \left| \frac{\partial a_{ij}(x,t,u)}{\partial u} \right| \leq \frac{\alpha e \sqrt{3}}{12mC_0}. \quad (8)$$

If the functions $a_{ij}(x, t, u)$, $a_i(x, t, u)$, $a(x, t, u)$, and $a_u(x, t, u)$ are periodic in t with period T , and for $(x, t) \in \bar{Q}$ and $|u| \leq C_0$ are continuous functions of (x, t, u) , having bounded derivatives with respect to x, u up to and including the fourth order, satisfying a Hölder condition in x, u , then in \bar{Q} there exists a continuous function $u(x, t)$, periodic in t with period T , possessing inside Q

with continuous derivatives entering equation (2), satisfying equation (2) inside Q , and on S the condition

$$u|_S = 0. \quad (9)$$

The scheme of proof of Theorem 4 is the same as that of Theorem 1. To prove the solvability of boundary-value problems of the form (3), (4) for nonlinear elliptic systems, the Leray-Schauder theorem is used. Verification of the conditions of this theorem is based mainly on the a priori estimates established by Schauder and Caccioppoli¹¹ for the solution of the first boundary-value problem for a linear elliptic equation. The smoothness inside Q of the solutions of the above-mentioned elliptic systems, necessary for obtaining further estimates, is established with the help of Schauder's results¹².

The remark made above concerning methods for obtaining estimates that guarantee the convergence of solutions of finite-difference problems to the solution of the original problem remains valid also in the case under consideration. It is obvious that the results formulated above are automatically carried over to backward parabolic equations.

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