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Abstract

Full Text

MATHEMATICS

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ON THE CONVERGENCE OF THE RESIDUAL OF THE BUBNOV-GALERKIN AND RITZ METHODS

(Presented by Academician V. I. Smirnov on 20 VI 1961)

In the paper ⁽¹⁾ it was proved that the residual of the Ritz method for the equation $Au = f$, where A is a positive-definite, self-adjoint operator, converges to zero if the operator has a purely point spectrum and the system of eigen-elements of a convergent operator is taken as the coordinate elements.

In the present paper this result is generalized.

1°. In a separable Hilbert space H consider the linear equation

$$Lu \equiv Au + Ku = f. \tag{1}$$

The operator A , with domain of definition $D(A)$, is assumed to be positive-definite and self-adjoint. We shall suppose that $D(A) \subset D(K)$.

Let A and B be convergent ⁽¹⁾ operators, i.e., both are positive-definite, self-adjoint, and $D(A) = D(B)$. Then $D(A^\alpha) = D(B^\alpha)$ ($0 \leq \alpha \leq 1$) ⁽²⁾, and, in particular, from the condition that the domains of definition coincide it follows that $H_A = H_B$ ⁽³⁾. If the operators A and B are convergent, then the operators AB^{-1} , BA^{-1} , $\sqrt{A}\sqrt{B^{-1}}$, $\sqrt{A^{-1}}$, \sqrt{B} , and the operators obtained from them by permuting the factors are bounded; hence the inequality immediately follows

$$d_1 \leq \frac{\|u\|_A}{\|u\|_B} \leq d_2, \quad d_1, d_2 > 0. \tag{2}$$

Let $\{\lambda_n\}$ be a sequence of positive numbers, and let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of elements such that

$$B\varphi_n = \lambda_n\varphi_n + \psi_n, \quad n = 1, 2, \dots$$

We assume that $\{\varphi_n\}$ is a system of elements in H_B that is orthonormal in the metric of the space H and complete (from inequality (2) its completeness in the space H_A follows).

Next suppose that:

1.
$$\sum_{i,j=1}^n (B\varphi_i, \varphi_j) t_i \bar{t}_j \geq \frac{1}{2} \sum_{i=1}^n \lambda_i |t_i|^2 \quad \text{for any } n \ (n = 1, 2, \dots).$$

2.
$$\sum_{i=1}^{\infty} \left\{ \frac{\|\psi_i\|}{\lambda_i} \right\}^2 < \frac{d_1^4}{4\|AB^{-1}\|^2}.$$

3. The operator KA^{-1} is completely continuous.

For solving equation (1) we shall use the Bubnov-Galerkin method, choosing the elements $\{\varphi_n\}$ as coordinate elements.

The aim of the present note is to prove that, under the stated assumptions, the relation

$$\|Lu_n - f\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

holds, where u_n are the approximations constructed by the Bubnov-Galerkin method. We have

$$u_n = \sum_{k=1}^n a_k \varphi_k.$$

The coefficients a_k are determined from the algebraic system

$$\sum_{k=1}^n a_k (A\varphi_k, \varphi_m) + \sum_{k=1}^n a_k (K\varphi_k, \varphi_m) = (f, \varphi_m), \quad m = 1, 2, \dots, n,$$

which is easily reduced to the form

$$\begin{aligned} & \sum_{k=1}^n b_k (\varphi_k, B^{-1}A\varphi_m) + \sum_{k=1}^n b_k \left(\frac{\psi_k}{\lambda_k}, B^{-1}A\varphi_m \right) + \\ & + \sum_{k=1}^n b_k (BA^{-1}KB^{-1}\varphi_k, B^{-1}A\varphi_m) + \sum_{k=1}^n b_k \left(BA^{-1}KB^{-1} \frac{\psi_k}{\lambda_k}, B^{-1}A\varphi_m \right) = \\ & = (BA^{-1}f, B^{-1}A\varphi_m), \quad m = 1, 2, \dots, n, \quad \text{where } b_k = \lambda_k a_k. \end{aligned} \quad (3)$$

Put $v_n = \sum_{k=1}^n b_k \varphi_k$ and consider the equation

$$v + BA^{-1}KB^{-1}v + T_1v + BA^{-1}KB^{-1}T_1v = BA^{-1}f, \quad (4)$$

where the operator

$$T_1v = \sum_{k=1}^{\infty} (v, \varphi_k) \frac{\psi_k}{\lambda_k}$$

is completely continuous by virtue of condition 2.

The identity $KB^{-1} = KA^{-1}AB^{-1}$ shows that the operator KB^{-1} is completely continuous. Hence the operator $BA^{-1}KB^{-1} + T_1 + BA^{-1}KB^{-1}T_1$ is also completely continuous.

As in paper ⁽¹⁾, it is proved that the relation $v_n \rightarrow v$ holds as $n \rightarrow \infty$, where v is the exact solution of equation (4), and the coefficients b_k in the representation $v_n = \sum b_k \varphi_k$ are determined from the algebraic system (3).

It is not difficult to see that the equality

$$Lu_n = AB^{-1}v_n + AB^{-1}(BA^{-1}KB^{-1}v_n + T_1v_n + BA^{-1}KB^{-1}T_1v_n),$$

holds, whose right-hand side tends, as $n \rightarrow \infty$, to the right-hand side of equation (1)—to the element f , and, consequently,

$$\|Lu_n - f\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

2°. In equation (1) put $K = 0$. Then the result obtained in item 1° shows that the residual $Au_n - f$ of the Ritz method for the equation $Au = f$ converges to zero, where A is a positive-definite self-adjoint operator and u_n are the Ritz approximations, provided only that the system of elements $\{\varphi_n\}$ of item 1° is taken as the coordinate system. The case when all $\psi_k = 0$, $k = 1, 2, \dots$, was considered by S. G. Mikhailin ⁽¹⁾: the residual of the Ritz method converges to zero if the operator has a purely point spectrum and the system of eigenfunctions of the convergent operator is taken as the coordinate system.

3°. In a separable Hilbert space H consider the equation

$$Lu \equiv Au + Ku = f, \quad (5)$$

without assuming that A is a positive-definite operator; we shall assume only that $D(A)$ is dense in H .

We shall suppose that $D(A) \subset D(K)$.

Let there exist sequences of positive numbers $\{\lambda_n\}$ and of elements $\{\varphi_n\}$ and $\{\psi_n\}$ such that

$$A\varphi_n = \lambda_n\varphi_n + \psi_n, \quad n = 1, 2, \dots,$$

where the system $\{\varphi_n\}$ is complete and orthonormal in the space H .

Next, suppose that the operator A^{-1} exists and that the operators

$$KA^{-1} \quad \text{and} \quad T_1 v = \sum_{k=1}^{\infty} (v, \varphi_k) \frac{\psi_k}{\lambda_k}$$

are completely continuous in the space H . Finally, assume that equation (5) is uniquely solvable.

We shall construct approximate solutions of equation (5) in the form $u_n = \sum_{k=1}^n a_k \varphi_k$, and determine the coefficients a_k from the system of algebraic equations

$$(Lu_n, \varphi_m) = (f, \varphi_m), \quad m = 1, 2, \dots, n. \quad (6)$$

We shall prove that, for sufficiently large n , system (6) is uniquely solvable and that the relation $\|Lu_n - f\| \rightarrow 0$ holds as $n \rightarrow \infty$. Denote

$$v_n = \sum_{k=1}^n b_k \varphi_k, \quad b_k = \lambda_k a_k.$$

The system of equations (6) can be rewritten as

$$(v_n + Tv_n, \varphi_m) = (f, \varphi_m), \quad m = 1, 2, \dots, n, \quad (7)$$

where $T = KA^{-1} + T_1 + KA^{-1}T_1$ is a completely continuous operator. Consider the equation

$$v + Tv = f. \quad (8)$$

It can be proved that, for sufficiently large n , system (7) is uniquely solvable and that the relation $\|v_n - v\| \rightarrow 0$ holds as $n \rightarrow \infty$, where v satisfies equation (8) ⁴.

It can also be shown that the equality $Lu_n = v_n + Tv_n$ holds, whose right-hand side tends to the element f , the right-hand side of equations (5) and (8), and, consequently, $\|Lu_n - f\| \rightarrow 0$ as $n \rightarrow \infty$.

Consider the equation

$$v + Tv = f. \quad (9)$$

It can be proved that, for sufficiently large n , system (7) is uniquely solvable and that the relation $v_n \rightarrow v$ holds as $n \rightarrow \infty$, where v satisfies equation (8) ⁴.

It can also be shown that the equality $Lu_n = v_n + Tv_n$ holds, whose right-hand side tends to the element f , the right-hand side of equations (5) and (8), and, consequently, $\|Lu_n - f\| \rightarrow 0$ as $n \rightarrow \infty$.

Remark. If the operator $(A+K)^{-1}$ or A^{-1} is bounded, then also $\|u_n - u_0\| \rightarrow 0$ as $n \rightarrow \infty$, where $u_0 = (A+K)^{-1}f$.

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Note: Figure translations are in progress. See original paper for figures.

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