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Abstract

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THEORY OF ELASTICITY

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LINEARIZATION OF THE EQUATIONS OF THE PLANE PROBLEM IN THE THEORY OF AN IDEALLY PLASTIC BODY

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As is known ([2]), the plane stress and strain states of an ideally plastic body are described by means of two equilibrium equations

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = 0, \qquad \frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} = 0, \tag{1}$$

to which it is necessary to add the plasticity conditions used. To solve the problem, we shall take, respectively, as the initial plasticity conditions the Saint-Venant and Mises conditions

$$(X_x - Y_y)^2 + 4X_y^2 = 4k_0^2, \qquad X_x^2 - X_{xY}y + Y_y^2 + 3X_y^2 = 4k_1^2 \tag{2}$$

and the generalized Mises-Schleicher plasticity condition for the plane strain state

$$(X_x - Y_y)^2 + 4X_y^2 = 4k_0^2 f_0 \left[{}^1/_2(X_x + Y_y)\right], \tag{3}$$

where X_x, Y_y, X_y are the components of stress; k_0, k_1 are constants of plasticity; f_0 is a positive function of the half-sum of the normal stresses.

The usual method of linearization of system (1), adopted in the literature, under conditions (2) and (3), is based on identically satisfying the plasticity conditions and subsequently transforming system (1) ([1-3]). The method proposed below is based on using the stress function introduced by Saint-Venant into the theory of plasticity.

If the solution of the original equations is taken in the form

$$X_x = \frac{\partial^2 F}{\partial y^2}, \qquad Y_y = \frac{\partial^2 F}{\partial x^2}, \qquad X_y = -\frac{\partial^2 F}{\partial x\,\partial y},$$

then equations (1) will be identically satisfied, while conditions (2) and (3) will each provide one equation for determining the function F for each of the plasticity conditions.

For the Saint-Venant plasticity condition we obtain the equation

$$\left(\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2}\right)^2 + 4\left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2 = 4k_0^2. \tag{4}$$

For the Mises condition and the generalized condition (3), the analogous equations will be

$$\left(\frac{\partial^2 F}{\partial y^2}\right)^2 - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \left(\frac{\partial^2 F}{\partial x^2}\right)^2 + 3\left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2 = 4k_1^2, \tag{5}$$

$$\left(\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x^2}\right)^2 + 4\left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2 = 4k_0^2 f_0\left[\frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)\right]. \tag{6}$$

1. Linearization of equations (4) and (6). We shall carry out the simplification of these equations in the complex domain. We shall proceed from the equa-

···of equation (6), which, after replacing the variables x, y by the complex variables $z = x + iy$, $\bar{z} = x - iy$, takes the form

$$r = f(s) : t, \tag{7}$$

where

$$r = \frac{\partial^2 F^*}{\partial \bar{z}^2}, \qquad t = \frac{\partial^2 F^*}{\partial z^2}, \qquad s = \frac{\partial^2 F^*}{\partial z\,\partial \bar{z}}, \qquad F^* = \frac{2}{k_0}F, \qquad f(s) = f_0(k_0 s). \tag{8}$$

Differentiating equation (7) with respect to $\bar{z}$ and introducing the new function $u = \partial F^*/\partial z$, we arrive at the equation

$$\left(\frac{\partial u}{\partial z}\right)^2 \frac{\partial^2 u}{\partial \bar{z}^2} - f_s'\left(\frac{\partial u}{\partial \bar{z}}\right)\frac{\partial^2 u}{\partial z\,\partial \bar{z}} + f\left(\frac{\partial u}{\partial \bar{z}}\right)\frac{\partial^2 u}{\partial z^2} = 0, \tag{9}$$

which in structure resembles the equation of the plane problem of gas dynamics (4).

We transform (9) by means of the Legendre substitution

$$u = -\Phi + \eta \frac{\partial \Phi}{\partial \eta} + \theta \frac{\partial \Phi}{\partial \theta}, \qquad \eta = \frac{\partial u}{\partial z}, \qquad \theta = \frac{\partial u}{\partial \bar{z}}, \tag{10}$$

thereby introducing the plane of variables η, θ, which, by virtue of its physical meaning, we shall call the stress plane. Equation (9) in the variables of this plane takes the form

$$\theta^2 \frac{\partial^2 \Phi}{\partial \theta^2} + f'(\eta) \frac{\partial^2 \Phi}{\partial \eta \, \partial \theta} + f(\eta) \frac{\partial^2 \Phi}{\partial \eta^2} = 0. \tag{11}$$

For each prescribed function f, the form of this linear equation, after reducing it to canonical form, can be substantially simplified.

If $f = 1$, i.e., for the Saint-Venant plasticity condition, equation (11) will be

$$\theta^2 \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial \eta^2} = 0. \tag{12}$$

Transforming this equation by means of the substitutions $\theta_1 = \ln \theta$, $\Phi = e^{1/2\,\theta_1} \Phi_1$, $\alpha = \eta + i\theta_1$, $\beta = \eta - i\theta_1$, we obtain the telegraph equation of the form

$$\frac{\partial^2 \Phi_1}{\partial \alpha \, \partial \beta} - \frac{1}{16} \Phi_1 = 0. \tag{13}$$

Let us carry out the linearization of equation (4) in the plane of real variables. Equations linearized in this domain are of interest, since such equations, even if structurally complex, can easily be reduced to a form convenient for programming and for solving the posed problem by means of high-speed computing machines.

Solving equation (4) with respect to $\partial^2 F / \partial x^2$ and differentiating the left- and right-hand sides with respect to y, we find

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + 2\varkappa \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} : \sqrt{k_0^2 - \left(\frac{\partial u}{\partial x}\right)^2} = 0, \qquad u = \frac{\partial F}{\partial y}, \qquad \varkappa = \pm 1. \tag{14}$$

In the variables of the stress plane this equation will be linear and is written in the form

$$\frac{\partial^2\Phi}{\partial\theta^2} - \frac{\partial^2\Phi}{\partial\eta^2} + 2\varkappa\eta\frac{\partial^2\Phi}{\partial\theta\,\partial\eta} : \sqrt{k_0^2 - \eta^2} = 0. \tag{15}$$

A simplification of equation (6) in the real domain can be carried out if the form of the function f is specified, for example, in the form of a polynomial of the second degree.

2. Linearization of equation (5). In the complex variables z and $\bar{z}$ this equation is written in the form

$$r = 1 : t - s^2 : 4t, \tag{16}$$

where r, t, s are determined according to formulas (8), in which $F^* = \dfrac{2}{k_1}F$.

Differentiating (16) with respect to $\bar{z}$, we find the equation

$$\left(\frac{\partial u}{\partial z}\right)^2 \frac{\partial^2 u}{\partial z^2} + \frac{1}{2}\frac{\partial u}{\partial z}\frac{\partial u}{\partial \bar{z}}\frac{\partial^2 u}{\partial z\,\partial \bar{z}} + \left[1 - \frac{1}{4}\left(\frac{\partial u}{\partial z}\right)^2\right]\frac{\partial^2 u}{\partial \bar{z}^2} = 0, \qquad u = \frac{\partial F^*}{\partial \bar{z}},$$

which in the variables of the stress plane has the form

$$\theta^2\frac{\partial^2\Phi}{\partial\theta^2} - \frac{1}{2}\eta\theta\frac{\partial^2\Phi}{\partial\eta\,\partial\theta} + \left[1 - \frac{1}{4}\eta^2\right]\frac{\partial^2\Phi}{\partial\eta^2} = 0.$$

Transforming this equation by means of the substitutions

$$\theta_1 = \ln\theta, \qquad \eta = 2\sin\eta_1, \qquad \Phi = e^{1/2\theta_1}\Phi_1,$$

we obtain

$$\frac{\partial^2\Phi_1}{\partial\theta_1^2} + \frac{1}{4}\frac{\partial^2\Phi_1}{\partial\eta_1^2} - \frac{1}{2}\operatorname{tg}\eta_1\frac{\partial^2\Phi_1}{\partial\eta_1\partial\theta_1} - \frac{1}{4}\Phi_1 = 0.$$

Carrying out the linearization of equation (5) in the real domain, we obtain the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u}{\partial y^2} + \frac{3\chi}{\sqrt{4k_1^2 - \frac{3}{4}(\partial u/\partial y)^2 - 3(\partial u/\partial x)^2}}\times$$

$$\times\left[\frac{1}{4}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\,\partial y}\right] = 0, \qquad u = \frac{\partial F}{\partial y}, \qquad \chi = \pm 1,$$

which becomes linear in the variables of the stress plane:

$$\frac{\partial^2\Phi}{\partial\theta^2} - \frac{1}{2}\frac{\partial^2\Phi}{\partial\eta^2} + \frac{3\chi}{\sqrt{4k_1^2 - \frac{3}{4}\theta^2 - 3\eta^2}}\left[\frac{1}{4}\theta\frac{\partial^2\Phi}{\partial\theta^2} - \eta\frac{\partial^2\Phi}{\partial\eta\,\partial\theta}\right] = 0.$$

Let us note that, with the help of the equations proposed by us, a closer connection may be established between the theory of plasticity and problems of gas dynamics. Certain facts known in hydrodynamics can apparently be transferred to the theory of plasticity.

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Note: Figure translations are in progress. See original paper for figures.