# Mathematics

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## Abstract

## Full Text

*Mathematics*

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# ON THE BEST UNIFORM APPROXIMATION

# ON CERTAIN CLASSES OF CONTINUOUS FUNCTIONS

*(Presented by Academician A. N. Kolmogorov, 12 V 1961)*

**1°.** We shall denote by $KH^{(\alpha)}$ the class of functions $f(t)$ of period $2\pi$ satisfying, on the entire real axis, the Lipschitz condition

$$|f(t') - f(t'')| \leq K|t' - t''|^{\alpha}, \qquad (0 < \alpha \leq 1). \tag{1}$$

The class of functions $f$ satisfying condition (1) on the interval $[-1, 1]$ will be denoted by $KH^{(\alpha)}_{[-1,1]}$.

Let $E_n(f)$ be the best uniform approximation of a periodic function $f$ by trigonometric polynomials of order not exceeding $n$; $E_n(f; -1, 1)$ the best approximation of the function $f$ by algebraic polynomials of degree not exceeding $n$ on the interval $[-1, 1]$.

Favard ([1]) and, simultaneously with him, N. I. Akhiezer and M. G. Krein ([2]) obtained exact expressions for the upper bounds of the best approximations of periodic functions having a bounded derivative of order $r$. In particular, they showed that

$$\sup_{f \in KH^{(1)}} E_n(f) = \frac{K\pi}{2(n+1)}, \qquad n = 0, 1, 2, \ldots. \tag{2}$$

For the case $0 < \alpha < 1$, Favard indicated the obvious lower estimate:

$$\sup_{f \in KH^{(\alpha)}} E_n(f) \geq \frac{K\pi^{\alpha}}{2(n+1)^{\alpha}}. \tag{3}$$

S. M. Nikol' skii ([3]) established that

$$\sup_{f \in KH^{(1)}_{[-1,1]}} E_n(f; -1, 1) = \frac{K\pi}{2(n+1)} - \varepsilon_n,$$

where $\varepsilon_n > 0,\ \varepsilon_n = O\left(\frac{1}{n \lg n}\right)$.

Generalizing this result, S. N. Bernstein ([4]) showed that, if (without loss of generality one may put $K = 1$)

$$\sup_{f \in H^{(\alpha)}_{[-1,1]}} E_n(f; -1, 1) = \frac{c_n(\alpha)}{(n+1)^\alpha}, \qquad \sup_{f \in H^\alpha} E_n(f) = \frac{\tilde{c}_n(\alpha)}{(n+1)^\alpha}, \qquad n = 0, 1, 2, \dots,$$

then for $0 < \alpha \leq 1$

$$c_n(\alpha) \leq \tilde{c}_n(\alpha) \leq \lim_{n\to\infty} c_n(\alpha) = \lim_{n\to\infty} \tilde{c}_n(\alpha) = c(\alpha). \tag{4}$$

From (3), or from the fact that $c_0(\alpha) = \pi^\alpha/2$, in view of (4) it followed that $c(\alpha) \geq \pi^\alpha/2$. The exact value of the constant $c(\alpha)$ for $0 < \alpha < 1$ remained unknown.

We shall prove that $c(\alpha) = \pi^\alpha/2$. This will immediately follow from the following assertion:

**Theorem.** For all $n = 0, 1, 2, \dots$ and $0 < \alpha \leqslant 1$,

$$\sup_{f \in H^{(\alpha)}} E_n(f) = \frac{\pi^\alpha}{2(n+1)^\alpha}. \tag{5}$$

Taking (2) and (3) into account, it is enough to establish that, for $0 < \alpha < 1$,

$$\sup_{f \in H^{(\alpha)}} E_n(f) \leqslant \frac{\pi^\alpha}{2(n+1)^\alpha}. \tag{6}$$

The proof of inequality (6) is based on the following lemmas.

**Lemma 1.** Let $f(t) \in H^{(\alpha)}$, and let $f_1(t)$ be a polygonal line inscribed in $f$ in an arbitrary way. Then $f_1 \in H^{(\alpha)}$.

The proof of Lemma 1, based on the fact that the function $t^\alpha$ is concave downward for $t > 0$, presents no special difficulties.

Let us consider periodic functions satisfying condition (1) on the whole axis and whose graph is a polygonal line with a finite number of vertices on the period $[0, 2\pi)$. We shall denote the class of such functions by $Kh^{(\alpha)}$.

**Lemma 2.** For all $n = 0, 1, 2, \dots,\ 0 < \alpha \leqslant 1$,

$$\sup_{f \in H^{(\alpha)}} E_n(f) = \sup_{f \in h^{(\alpha)}} E_n(f). \tag{7}$$

Since $h^{(\alpha)} \subset H^{(\alpha)}$, we have

$$\sup_{f\in h^{\alpha}} E_n(f) \leqslant \sup_{f\in H^{(\alpha)}} E_n(f). \tag{8}$$

On the other hand, let $f \in H^{(\alpha)}$, and let $T_n(f,t)$ be the trigonometric polynomial of best approximation to the function $f$. Then on $[0,2\pi)$ there are $2n+2$ points $x_1, x_2, \dots, x_{2n+2}$ at which the difference $f - T_n$ takes the values $\pm E_n(f)$, alternately changing sign. Define a continuous periodic function $f_1(t)$ by setting $f_1(x_k) = f(x_k)$ and taking $f_1(t)$ to be linear on the intervals $(x_k, x_{k+1})$. By Lemma 1, $f_1 \leqslant h^{(\alpha)}$, and the well-known theorem of Valle-Poussin (see, for example, [5]) allows us to conclude that $E_n(f_1) \geqslant E_n(f)$, and consequently

$$\sup_{f\in h^{(\alpha)}} E_n(f) \geqslant \sup_{f\in H^{(\alpha)}} E_n(f). \tag{9}$$

From (8) and (9), (7) follows.

**Lemma 3.** For arbitrary $K > 0,\ 0 < \alpha < 1$, for every function $f \in h^{(\alpha)}$ there exists a function $\varphi_0 \in Kh^{(1)}$ such that

$$\|f - \varphi_0\| = \max_t |f(t) - \varphi_0(t)| \leqslant \frac{1-\alpha}{2}\left(\frac{K}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}. \tag{10}$$

If $f \in Kh^{(1)}$, then the lemma is trivial; therefore we assume that $f \notin Kh^{(1)}$, i.e. the polygonal line $f(t)$ has at least one segment whose angular coefficient is, in absolute value, greater than $K$.

Let $x_1, x_2, \dots, x_q$ be the abscissas of the vertices of the polygonal line $f$ on $[0,2\pi)$, and let $M$ be the set of all functions of the class $Kh^{(1)}$ having, on $[0,2\pi]$, vertices at the points with abscissas $x_k,\ k = 1,2,\dots,q$, and only at these points. Let $\varphi_0$ be that function in $M$ for which

$$\|f - \varphi_0\| = \inf_{\varphi\in M} \|f - \varphi\| = \rho, \tag{11}$$

and, since $f \notin Kh^{(1)}$, $\rho > 0$. The validity of Lemma 3 will be estab-

established if we show that

$$\rho \leq \frac{1-\alpha}{2}\left(\frac{K}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

The proof of this inequality is based on the following circumstance: among the links of the broken line $\varphi_0$ there is at least one link $l$, with abscissas of the endpoints $x_k$ and $x_m$, satisfying the following two conditions: 1) the angular coefficient $\gamma$ of the link $l$ is equal in absolute value to $K$; 2) $\varphi_0(x_k) - f(x_k) = f(x_m) - \varphi_0(x_m) = \rho \operatorname{sign}\gamma$.

Indeed, assuming that the broken line $\varphi_0$ has no such link, one can show the possibility of constructing a function $\varphi_1 \in M$ for which $\|f - \varphi_1\| < \rho$, which is impossible in view of (11).

But if for the link $l$ on the interval $(x_k, x_m)$ conditions 1) and 2) are fulfilled, then

$$|f(x_k) - f(x_m)| = 2\rho + K|x_k - x_m|.$$

Hence

$$\rho \le \frac{1}{2}\{|x_k - x_m|^\alpha - K|x_k - x_m|\} \le \frac{1}{2}\max_{t>0}(t^\alpha - Kt) = \frac{1-\alpha}{2}\left(\frac{K}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

Now it is easy to prove (6). Let $f \in h^{(\alpha)}$, and for any fixed $K > 0$, let $\varphi_0$ be that function from the class $Kh^{(\alpha)}$ for which (10) holds. If $T_n(\varphi_0, t)$ is the trigonometric polynomial of best approximation for the function $\varphi_0 \in Kh^{(1)}$, then, by (2),

$$E_n(t) \le \|f(t) - T_n(\varphi_0, t)\| \le \|f - \varphi_0\| + \|\varphi_0 - T_n(\varphi_0, t)\| \le$$

$$\le \frac{1-\alpha}{2}\left(\frac{K}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} + \frac{K\pi}{2(n+1)} = \psi(K).$$

Since this is valid for every $f \in h^{(\alpha)}$ and arbitrary $K > 0$, in view of (7),

$$\sup_{f\in H^{(\alpha)}} E_n(f) = \sup_{f\in h^{(\alpha)}} E_n(l) \le \min_{K>0}\psi(K) = \frac{1}{2}\left(\frac{\pi}{n+1}\right)^\alpha.$$

2°. Result (5) generalizes to the class $H_\omega$ of continuous periodic functions whose modulus of continuity does not exceed a given modulus of continuity $\omega(t)$, under the additional assumption that the function $\omega(t)$ is convex upward for $t > 0$. In this case the same method of reasoning makes it possible to establish that

$$\sup_{f\in H_\omega} E_n(f) = \frac{1}{2}\omega\left(\frac{\pi}{n+1}\right). \tag{12}$$

Indeed, Lemmas 1 and 2 are formulated and proved in a completely analogous way; $h_\omega$ is defined as the subclass of functions of the class $H_\omega$ which are broken lines with a finite number of nodes on the period. Then in Lemma 3 one may immediately choose $K$ satisfying the inequalities

$$\omega'_+\left(\frac{\pi}{n+1}\right) \le K \le \omega'_-\left(\frac{\pi}{n+1}\right),$$

and, using the fact that for such a choice of $K$

$$\max_{t>0}[\omega(t) - Kt] = \omega\left(\frac{\pi}{n+1}\right) - K\frac{\pi}{n+1},$$

by the same method prove the existence of a function $\varphi_0 \in Kh^{(1)}$ for which

$$\|f - \varphi_0\| \leq \frac{1}{2}\left\{\omega\left(\frac{\pi}{n+1}\right) - K\frac{\pi}{n+1}\right\}^{*}.$$

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* Since into every function $f \in H_\omega$ one can inscribe $f_1 \in h_\omega$ so that $\|f - f_1\| < \varepsilon$, the preceding arguments make it easy to conclude that

$$\sup_{f\in H_\omega} \inf_{\varphi\in KH^{(1)}} \|f - \varphi\| = \frac{1}{2}\max_{t>0}[\omega(t) - Kt].$$

3°. Let $U_n$ be a linear operator by means of which to each periodic integrable function $f$ there is assigned the trigonometric polynomial

$$U_n(f, x, \lambda) = \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_k^{(n)}(a_k \cos kx + b_k \sin kx), \tag{13}$$

where $a_0,\ a_k,\ b_k,\ k = 1, 2, \dots, n$, are the Fourier coefficients of the function $f$, and $\lambda_k^{(n)}$ are material multipliers.

If $f_0 \in H^{(\alpha)}$ is an even function with period $\dfrac{2\pi}{n+1}$, defined on $\left[0, \dfrac{\pi}{n+1}\right]$ by the equality $f_0(t) = t^\alpha$, then, expanding $f_0$ in a Fourier series, it is easy to compute that, whatever the $\lambda_k^{(n)}$ may be, for all $n = 0, 1, 2, \dots$,

$$\sup_{f\in H^{(\alpha)}} \|f - U_{nf}\| \geq |U_n(f_0, 0, \lambda)| = \frac{\pi^\alpha}{(1+\alpha)(n+1)^\alpha} \qquad (0 < \alpha \leq 1).$$

This, in comparison with (5), shows that for $0 < \alpha \leq 1$, for every linear operator of the form (13),

$$\sup_{f\in H^{(\alpha)}} \|f - U_{nf}\| \geq \sup_{f\in H^{(\alpha)}} E_n(f) + \frac{1-\alpha}{1+\alpha}\left(\frac{\pi}{n+1}\right)^\alpha.$$

By means of analogous reasoning it is easily proved that if the function $\omega(t)$ is convex upward and is not linear on the interval $\left(0, \dfrac{\pi}{n+1}\right)$, then

$$\inf_{\lambda} \sup_{f\in H_\omega} \|f - U_{nf}\| \geq \frac{n+1}{\pi} \int_0^{\frac{\pi}{n+1}} \omega(t)\,dt > \frac{1}{2}\omega\left(\frac{\pi}{n+1}\right) = \sup_{f\in H_\omega} E_n(f),$$

i.e., there does not exist a linear operator (13) which would realize on the class $H_\omega$ the upper bound of the best approximations*.

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*Note: Figure translations are in progress. See original paper for figures.*