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Abstract

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ON THE OUTFLOW OF A RAREFIED GAS INTO A VACUUM FROM A POINT SOURCE

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In this note the propagation in a vacuum of particles of a sufficiently rarefied gas from a point source*, situated in the plane $z = 0$, is considered from the standpoint of solving an equation of the Einstein–Fokker type ([1]), with a time-dependent "diffusion coefficient," under boundary conditions corresponding to different types of interaction of the gas particles with the plane $z = 0$.

1. In the absence of the plane $z = 0$, the probable departure velocities u for an individual particle are specified by the Maxwell distribution for the gas in the source

$$p = \frac{4}{\sqrt{\pi c^3}}\, u^2 \exp\left[-\frac{u^2}{c^2}\right] du, \tag{1}$$

where $c = \sqrt{2kT/m}$ is the most probable particle velocity at the given gas temperature T; m is the molecular weight of the particle; k is Boltzmann' s constant. In other words, p is the probability that a particle which has flown out of a source placed at the point $x = y = z = 0$ will, per unit time, be found in a spherical layer of radius u and thickness du. Obviously, after a time t the particle will be found in a spherical layer of radius $r = ut$ and thickness $dr = t\, du$ with the same probability

$$p = \frac{4}{\sqrt{\pi}\, c^3 t^3}\, r^2 \exp\left[-\frac{r^2}{c^2 t^2}\right] dr. \tag{2}$$

Dividing (2) by the volume of the spherical layer $4\pi r^2 dr$, we obtain the probability that the particle under consideration will, after a time t, enter a unit volume located at the point (x, y, z); $r = \sqrt{x^2 + y^2 + z^2}$:

$$q_{1t} = \frac{1}{\pi\sqrt{\pi}\, c^3 t^3} \exp\left[-\frac{x^2 + y^2 + z^2}{c^2 t^2}\right]. \tag{3}$$

2. However, the presence of the plane $z = 0$ directly behind the source will change the probability that a particle will reach a unit volume at the point

(x, y, z) of the empty half-space $z > 0$. It is clear that, for absolutely elastic reflection of the gas particles from the plane $z = 0$, probability (3) should be doubled,

$$q_{1t}^{\mathrm{e}} = \frac{2}{\pi\sqrt{\pi}\, c^3 t^3} \exp\left[-\frac{x^2 + y^2 + z^2}{c^2 t^2}\right]. \tag{4}$$

If at the time $t = 0$ N particles were released and, moreover, all the particles have an ordered macroscopic velocity U, constant

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* A source of particles is meant for which all directions of departure may be regarded as equally probable. It is assumed that the particles practically do not collide with one another.

in magnitude and directed along the axis OZ, then instead of (4) one should write

$$q_t = \frac{2N}{\pi\sqrt{\pi}\, c^3 t^3} \exp\left[-\frac{x^2 + y^2 + (z - Ut)^2}{c^2 t^2}\right]. \tag{5}$$

This geometrically obvious expression is a solution of the Einstein–Fokker type equation for the probability that the particles reach, at a specified time t, a unit volume located at the point (x, y, z):

$$\frac{\partial}{\partial t} q_t + U \frac{\partial}{\partial z} q_t = D \Delta q_t \tag{6}$$

with diffusion coefficient $D = c^2 t/2$, satisfying the initial condition

$$q_t\big|_{t=0} = N\delta(x)\delta(y)\delta(z) \tag{7}$$

and the boundary conditions

$$q_t \to 0 \quad \text{as } r \to \infty; \tag{8}$$

$$\left[D \frac{\partial}{\partial z} q_t - U q_t\right]_{z=0} = 0. \tag{9}$$

Condition (7) denotes an instantaneous point source at the origin, while condition (9) denotes elastic reflection of the particles from the plane $z = 0$.

In this interpretation, the instantaneous distribution of the probability of flight of particles through a unit area of the plane $z = z^*$ per unit time is written as the flux of the volume density of particles through the plane $z = z^*$

$$n_t\big|_{z=z^*} = \left[-D\frac{\partial q_t}{\partial z} + Uq_t\right]_{z=z^*} = \frac{2Nz^*}{\pi\sqrt{\pi}\,c^3t^4}\exp\left[-\frac{x^2+y^2+(z-Ut)^2}{c^2t^2}\right]. \quad (10)$$

Let us note that expression (10) can also be obtained directly by multiplying the volume concentration of particles (5) by the velocity at which they reach the plane $z = z^*$, equal to z^*/t.

If a stationary source of intensity N particles per unit time is placed at the origin, then the established distribution of the volume density of particles and of the flux density through the plane $z = z^*$ is found by integrating expressions (5) and (10) with respect to t from 0 to ∞:

$$q = \int_0^\infty q_t\,dt = \frac{N}{\pi cr^2}\exp\left[-\frac{U^2}{c^2}\left(1-\frac{z^2}{r^2}\right)\right] i^1\operatorname{erfc}\left(-\frac{Uz}{cr}\right); \quad (11)$$

$$n = \int_0^\infty n_t\,dt = \frac{2Nz^*}{\pi r^{*3}}\exp\left[\frac{U^2}{c^2}\left(1-\frac{z^{*2}}{r^{*2}}\right)\right] i^2\operatorname{erfc}\left(-\frac{Uz}{cr}\right), \quad (12)$$

where

$$r^* = \sqrt{x^2+y^2+z^{*2}};$$

$$\int_0^\infty t^{-m}\exp\left[+\frac{2Uz}{c^2t}-\left(\frac{r}{ct}\right)^2\right]dt =$$

$$= \left(\frac{c}{r}\right)^{m-1}\frac{\sqrt{\pi}}{2}(m-2)!\exp\left[+\left(\frac{Uz}{cr}\right)^2\right] i^{m-2}\operatorname{erfc}\left(-\frac{Uz}{cr}\right). \quad (13)$$

Since

$$i^n\operatorname{erfc}(0) = \frac{1}{2^n\Gamma(1+n/2)},$$

then for $U = 0$ ([2])

$$q = \frac{N}{\pi\sqrt{\pi}cr^2}; \quad (14)$$

$$n = \frac{Nz^*}{2\pi r^{*3}}. \quad (15)$$

3. In solving equation (6), in addition to the condition of reflection of particles from the plane $z = 0$, one may consider the case of absorption of gas particles on the plane $z = 0$, i.e., instead of imposing (9), require that

$$q_t\Big|_{z=0} = 0. \tag{16}$$

Conditions of this type are realized, for example, in the emission of charged particles from a source,* if the charged particle ceases to exist as such upon contact with the wall.

In the absence of a macroscopic velocity U, the solution of equation (6) is found as the superposition at the origin of coordinates of an instantaneous source and sink of equal intensity N, in the form of a point dipole

$$q_t = \frac{2N'z}{\pi\sqrt{\pi}\, c^5 t^5} \exp\left[-\frac{x^2+y^2+z^2}{c^2t^2}\right]. \tag{17}$$

Here $N' = \lim_{\substack{N\to\infty \\ l\to 0}} Nl$, and l is the vanishingly small distance between the source and the sink in the dipole. For $U \neq 0$ the required solution is obtained in the form of a series absolutely convergent as $1/\sqrt{k}$,

$$q_t = \frac{2N'z}{\pi\sqrt{\pi}\, c^5 t^5} \exp\left[-\frac{x^2+y^2+(z-Ut)^2}{c^2t^2}\right] \left\{1 + \sum_{k=1}^{\infty} (-1)^k b_k \left(\frac{z}{ct}\right)^k\right\}, \tag{18}$$

where

$$b_1 = \frac{U}{c}; \qquad b_2 = \frac{2}{3}\left(\frac{U}{c}\right)^2;$$

$$b_{k+1} = \frac{2(k-1)}{(k+1)(k+2)}\, b_{k-1} + \frac{2}{k+2}\,\frac{U}{c}\, b_k. \tag{19}$$

The corresponding instantaneous flux of particle density through the plane $z = z^*$ is written as

$$n_t = \left[-D\frac{\partial}{\partial z} q_t + U q_t\right]_{z=z^*} =$$

$$= \frac{2N'}{\pi\sqrt{\pi}\, c^3 t^4} \exp\left[-\frac{x^2+y^2+(z^*-Ut)^2}{c^2t^2}\right] \left\{\left(\frac{z^*}{ct}\right)^2 - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^k b_k \left[\left(\frac{z^*}{ct}\right)^{k+2} - \frac{k+1}{2}\left(\frac{z^*}{ct}\right)^k\right]\right\}. \tag{20}$$

In the case of a stationary dipole, the expressions for the volume density of particles and for the particle flux through a fixed plane $z = z^*$ can be obtained by integrating formulas (18) and (20) with respect to t from 0 to ∞. Taking (13) into account, we obtain

$$q = \frac{N'z}{\pi c r^4} \exp\left[-\frac{U^2}{c^2}\left(1 - \frac{z^2}{r^2}\right)\right]\{3!\, i^3 \operatorname{erfc}(-\upsilon) + $$

$$+ \sum_{k=1}^{\infty} (-1)^k b_k \left[\frac{z}{r}\right]^k (k+3)!\, i^{k+3} \operatorname{erfc}(-\upsilon)\Bigg\}, \tag{21}$$

* The particle density is assumed to be such that Coulomb forces may be neglected at a sufficiently small dist

$$n = \frac{N'}{\pi r^{*3}} \exp\left[-\frac{U^2}{c^2}\left(1 - \frac{z^{*2}}{r^{*2}}\right)\right]\left\{4!\, i^4 \operatorname{erfic}(-\upsilon^*)\left[\frac{z^*}{r^*}\right]^2 - \frac{1}{2} 2!\, i^2 \operatorname{erfic}(-\upsilon^*) + \right.$$

$$\left. + \sum_{k=1}^{\infty} (-1)^k b_k \left[(k+4)!\, i^{k+4} \operatorname{erfic}(-\upsilon^*)\left[\frac{z^*}{r^*}\right]^{k+2} - \frac{k+1}{2}(k+2)!\, i^{k+2} \operatorname{erfic}(-\upsilon^*)\left[\frac{z^*}{r^*}\right]^k\right]\right\}, \tag{22}$$

where $\upsilon = Uz/cr; \quad \upsilon^* = Uz^*/cr^*$.

For $U = 0$, (21) becomes

$$q = -\frac{N'z}{\pi\sqrt{\pi}\, c r^4}, \tag{23}$$

and (22) becomes

$$n = \frac{N'}{4\pi r^{*3}}\left[3\left(\frac{z^*}{r^*}\right)^2 - 1\right]. \tag{24}$$

Let us note that the flux (24) is positive everywhere inside the cone $2z^2 = x^2 + y^2$ and has the opposite sign outside it.

From the expressions obtained for the flux of particle density through the plane $z = z^*$, one can easily pass to the pressure distribution over the transverse section of the molecular flux ([3]).

The results obtained for a point source, integrated over the initial data, give the solution of the problem of efflux from a linear or plane source located in the plane $z = 0$.

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References

1. A. N. Kolmogorov, *UMN*, vol. 5 (1938).
2. O. S. Berlyand, L. A. Sokolovskaya, *Engineering Physics Journal*, 1, No. 11 (1958).
3. Kh. Sh. Tzyan, collection *Gas Dynamics*, IL, 1950.

Note: Figure translations are in progress. See original paper for figures.