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Abstract

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ON THE QUESTION OF THE LOCATION OF THE ZEROS OF DIRICHLET L-SERIES

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The present work is devoted to estimating sums of characters

$$\left|\sum_{k=T+1}^{T+N} \chi_D(k)\right|$$

(where $\chi_D(k)$ is a primitive character modulo D) and to their application to estimating the growth and the location of the zeros of L-series in the critical strip. The results of the present investigation develop the works of A. G. Postnikov (^{1}) and S. M. Rozin (^{2}).

§ 1. **Estimate of the sum of characters.**

Theorem 1. Let

$$S = \sum_{k=T+1}^{T+N} \chi_D(k),$$

where $\chi_D(k)$ is a primitive character modulo

$$D = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$$

with $\min(p_1, \dots, p_l) > 2$, and let $\alpha_\nu = \min(\alpha_1, \dots, \alpha_l)$ be such that

$$(\alpha_\nu / \log^3 \alpha_\nu)^{1/4} \geq 13 \quad (\alpha_\nu > 10^8,\ \log \alpha_\nu \geq 18.8),$$

and

$$\prod_{i=1}^{l} p_i^{3\alpha_i (\log^3 \alpha_\nu/\alpha_\nu)^{1/4}} < N.$$

Then the estimate

$$|S| < N \prod_{i=1}^{l} p_i^{-\frac{\alpha_i}{\alpha_\nu}(\alpha_\nu \log \alpha_\nu)^{1/4}}.$$

is valid.

Proof. Choose r from the equality

$$r = \left[\frac{\alpha_\nu}{(\alpha_\nu \log \alpha_\nu)^{3/4} - (\alpha_\nu \log \alpha_\nu)^{1/4}} \right].$$

It is clear that

$$r > \frac{\alpha_\nu}{(\alpha_\nu \log \alpha_\nu)^{3/4} - (\alpha_\nu \log \alpha_\nu)^{1/4}} - 1 > \left[\frac{\alpha_\nu}{\log^3 \alpha_\nu} \right]^{1/4} - 1 \geq 12.$$

Let

$$s_i = \left[\frac{\alpha_i + \delta}{r} \right] + 1,$$

where $\delta = -1$, if $\alpha_i \neq \alpha p_i^f - \nu$, $(\alpha, p_i) = 1$, $f = 1, 2, ...,$ $0 \leq \nu \leq f - 1$, and $\delta = \nu$, if $\alpha_i = \alpha p_i^f - \nu$. Then

$$s_i = \frac{\alpha_i - 1}{r} + \frac{s_{1i}}{r}, \qquad s_{1i} = d + \delta + 1 \quad (1 \leq d \leq r).$$

Denote by N_1 such a number that

$$N = \prod_{i=1}^{l} p_i^{2s_i - s_{1i}} N_1 + R.$$

$$s_i - s_{1i} = \alpha_i - 1 - s_i(r-1) = \alpha_i - 1 - \left[\frac{\alpha_i + \delta}{r} \right](r-1) - (r-1) \geq \frac{\alpha_i}{r} - \nu - r \geq 0,$$

since

$$r < \frac{2\alpha_\nu}{(\alpha_\nu \log \alpha_\nu)^{1/4}} < 2\alpha_\nu^{3/4}; \qquad \nu \leq f \leq \log_{p_i} \alpha_i < \log_3 \alpha_i,$$

$$R \leq \prod_{i=1}^{l} p_i^{2s_i - s_{1i}} \leq \prod_{i=1}^{l} p_i^{2s_i - 1}.$$

Then

$$\left|\sum_{k=T+1}^{T+N} \chi_D(k)\right| \leq N_1|S_1| + \prod_{i=1}^{l} p_i^{2s_i-1}.$$

Here

$$|S_1| = \max_{Q} \left| \sum_{k=Q+T+1}^{Q+T+\prod_{i=1}^{l} p_i^{2s_i-s_{1i}}} \chi_D(k) \right|.$$

But

$$\sum \chi_D(k) = \sum \chi_{p_1^{\alpha_1}}(k) \ldots \chi_{p_l^{\alpha_l}}(k) = \sum \exp\left\{2\pi i \sum_{i=1}^{l} \frac{m_i \operatorname{ind}_{g_i} k}{\varphi(p_i^{\alpha_i})}\right\},$$

where g_i is a primitive root modulo $p_i^{\alpha_i}$, and $(m_i, p_i) = 1$.
Let, for $Q = Q_1$, the maximum of the sum be attained

$$S_1 = \sum_{k=1}^{\prod_{i=1}^{l} p_i^{2s_i-s_{1i}}} \exp\left\{2\pi i \sum_{i=1}^{l} \frac{m_i \operatorname{ind}_{g_i}(k+T+Q_1)}{\varphi(p_i^{\alpha_i})}\right\}.$$

We make the substitution $k = x_1 + \prod_{i=1}^{l} p_i^{s_i} x_2$.

$$|S_1| \leq \sum_{x_1=1}^{\prod_{i=1}^{l} p_i^{s_i}} \left| \sum_{x_2=0}^{\prod_{i=1}^{l} p_i^{s-s_{1i}}} \exp\left\{2\pi i \sum_{i=1}^{l} \frac{m_i \operatorname{ind}_{g_i}\left(x_1+T+Q_1+\prod_{i=1}^{l} p_i^{s_i} x_2\right)}{\varphi(p_i^{\alpha_i})}\right\}\right|.$$

But it is known (¹) that if $(a, p) = 1$, and $aa' \equiv 1 \pmod{p^\nu}$, then

$$\operatorname{ind}_g(a + p^\mu u) \equiv \operatorname{ind}_g a + \lambda(p-1)f(a'u) \pmod{\varphi(p^n)},$$

where

$$f(u) \equiv u - \frac{p^\mu u^2}{2} + \ldots + \frac{(-1)^s p^{\mu s} u^{s+1}}{s+1} \pmod{p^{n-1}}, \qquad s = \left[\frac{n+\delta}{\mu}\right]$$

(δ was defined above).
Then

$$|S_1| \leq \sum_{x_1=1}^{\prod p_i^{s_i}} \left| \sum_{x_2=0}^{\prod p_i^{s_i - s_{1i}}} \exp\left\{ 2\pi i \left(\sum_{i=1}^{l} \frac{m_i \operatorname{ind}_{g_i}(x_1 + Q + T)}{\varphi(p_i^{\alpha_i})} + \right.\right.\right.$$

$$\left.\left.\left. + \sum_{i=1}^{l} \frac{m_i \lambda_i \left(c_i x_2 - \ldots + \frac{(-1)^{r-1}}{r} x_1^{r-1} (c_i x_2)^r p_i^{s_i(r-1)} \right)}{p_i^{\alpha_i - 1}} \right) \right\} \right|,$$

where $c_i = \prod_{j=1}^{l} p_j^{s_j}$, $i \neq j$,

$$|S_1| \leq \prod_{i=1}^{l} p_i^{s_i} |S_2|,$$

$$|S_2| = \left| \sum_{x_2=0}^{\prod_{i=1}^{l} p_i^{s_i - s_{1i}}} \exp\left\{ 2\pi i \sum_{i=1}^{l} \frac{a_{i1} x_2 - p_i^{s_{1i}} a_{i2} x_2^2 \frac{1}{2} + \ldots + \frac{(-1)^{r-1}}{r} p_i^{s_i(r-1)} a_{ir} x_2^r}{p_i^{s_i(r-1) + s_i - s_{1i}}} \right\} \right|,$$

where $(a_{ij}, p_i) = 1$, $j = 1, \ldots, l$; $i = 1, \ldots, l$.

Consider the coefficient of x_2^r. It is equal to

$$\frac{(-1)^{r-1}}{r} \sum_{i=1}^{l} \frac{a_{ir}}{p_i^{s_i - s_{1i}}} = \frac{(-1)^{r-1} A_r}{r \prod_{i=1}^{l} p_i^{s_i - s_{1i}}}, \qquad (A_r, p_i) = 1, \quad i = 1, \ldots, l;$$

$$r \prod_{i=1}^{l} p_i^{s_i - s_{1i}} < \left(\prod_{i=1}^{l} p_i^{s_i - s_{1i}} \right)^{r-1}.$$

Consequently, S_2 can be estimated by the estimate of I. M. Vinogradov (³)

$$|S_2| \leq r^{3r \log r} \prod_{i=1}^{l} p_i^{(s_i - s_{1i})\left(1 - \frac{1}{9r^2 \log r}\right)} = e^{3r \log^2 r} \prod_{i=1}^{l} p_i^{(s_i - s_{1i})\left(1 - \frac{1}{9r^2 \log r}\right)}.$$

Then

$$|S| \ll \prod_{i=1}^{l} p_i^{2s_i - 1} + N e^{3r \log^2 r} \prod_{i=1}^{l} p_i^{-(s_i - s_{1i})} \frac{1}{q^{r^2 \log r}},$$

$$\prod_{i=1}^{l} p_i^{2s_i - 1} < \frac{1}{3} \prod_{i=1}^{l} p_i^{2s_i} < \frac{1}{3} \prod_{i=1}^{l} p_i^{-\frac{\alpha_i}{\alpha_\nu} \sqrt[4]{\alpha_\nu \log \alpha_\nu}} N,$$

since

$$2s_i \le 2\left(\frac{\alpha_i+\delta}{r}+1\right) \le 2\left\{\frac{\dfrac{\alpha_i+\delta}{\alpha_\nu}}{(\alpha_\nu\log\alpha_\nu)^{3/4}-(\alpha_\nu\log\alpha_\nu)^{1/4}-1}+1\right\}$$

$$\ll 3\left\{\frac{\alpha_i}{\alpha_\nu}\left((\alpha_\nu\log\alpha_\nu)^{3/4}-(\alpha_\nu\log\alpha_\nu)^{1/4}+1\right)\right\} \le \frac{\alpha_i}{\alpha_\nu}\left(3(\alpha_\nu\log\alpha_\nu)^{3/4}-(\alpha_\nu\log\alpha_\nu)^{1/4}\right).$$

Now let us estimate the second term. It is easy to see that

$$\prod_{i=1}^{l} p_i^{-4r\log^2 r\,\frac{\alpha_i}{\alpha_\nu}} < \prod_{i=1}^{l} p_i^{-\sqrt[4]{\alpha_\nu\log\alpha_\nu}\,\frac{\alpha_i}{\alpha_\nu}-1} \ll \frac{1}{3}\prod_{i=1}^{l} p_i^{-\frac{\alpha_i}{\alpha_\nu}\sqrt[4]{\alpha_\nu\log\alpha_\nu}}.$$

Estimating (1) by means of this inequality, we obtain

$$|S| \le \frac{1}{3}\prod_{i=1}^{l} p_i^{-\frac{\alpha_i}{\alpha_\nu}\sqrt[4]{\alpha_\nu\log\alpha_\nu}}N + \frac{1}{3}\prod_{i=1}^{l} p_i^{-\frac{\alpha_i}{\alpha_\nu}\sqrt[4]{\alpha_\nu\log\alpha_\nu}} < N\prod_{i=1}^{l} p_i^{-\frac{\alpha_i}{\alpha_\nu}\sqrt[4]{\alpha_\nu\log\alpha_\nu}},$$

which was required to be proved.

§ 2. Growth and zeros of L-functions.

Theorem 2. Let $Q>3$ be a constant;

$$\log D \le \left(\frac{\alpha_\nu}{\log^3\alpha_\nu}\right)^{\frac{Q+1}{4}}; \quad D=p_1^{\alpha_1}\cdots p_l^{\alpha_l}; \quad s=\sigma+it; \quad |s|<C_1; \quad \sigma\ge 1-\frac{1}{(\log D)^{\frac{Q}{Q+1}}},$$

and let $\chi_D(k)$ be a primitive character modulo D. Then

$$|L(s,\chi)| < C(\log D)^{\frac{Q}{Q+1}}.$$

Lemma. If

$$\left|\sum_{x=1}^{k}\chi_D(x)\right| \ll \begin{cases} k, & \text{for } k\le D^\omega, \\ k\displaystyle\prod_{i=1}^{l} p_i^{\beta_i}, & \text{for } k> D^\omega, \end{cases} \qquad \sigma\ge 1-\gamma, \quad |s|<C_1,$$

and the conditions

1) $\omega\gamma \leq C_0/\log D$;
2) $\alpha_i\gamma \leq \beta_i, \qquad i = 1, 2, \dots, l,$

are satisfied, then

$$|L(s,\chi)| < C_2/\gamma.$$

Proof of the lemma. It is easy to see that

$$\sum_{k=1}^{N} a_{kb}k = \sum_{k=1}^{N-1} S_k(b_k - b_{k+1}), \qquad \text{where } S_k = \sum_{\nu=1}^{k} a_\nu, \quad S_0 = S_N = 0.$$

Choose $a_k = \chi_D(k)$, $b_k = 1/k^s$, $N = \lambda D$ (λ an integer). Then

$$\left|\sum_{k=1}^{\lambda D} \frac{\chi_D(k)}{k^s}\right| = \left|\sum_{k=1}^{\lambda D-1} S_k\left(\frac{1}{k^s} - \frac{1}{(k+1)^s}\right)\right| < c_1 \sum_{k=1}^{\lambda D} \frac{|S_k|}{k^{2-\gamma}};$$

$$\sum_{k=1}^{\lambda D} \frac{|S_k|}{k^{2-\gamma}} = \sum_{k=1}^{D} \frac{|S_k|}{k^{2-\gamma}} + \sum_{k_1=1}^{\lambda-1}\sum_{k_2=1}^{D} \frac{|S_{k_2}|}{(k_1 D + k_2)^{2-\gamma}} <$$

$$< \sum_{k=1}^{D} \frac{|S_k|}{k^{2-\gamma}} + \sum_{k_2=1}^{D} \frac{|S_{k_2}|}{D^{2-\gamma}}\left(\sum_{k_1=1}^{\lambda-1} \frac{1}{k_1^{2-\gamma}}\right) < c_3 \sum_{k=1}^{D} \frac{|S_k|}{k^{2-\gamma}}.$$

Hence

$$\left|\sum_{k=1}^{\lambda D} \frac{\chi_D(k)}{k^s}\right| < c_4 \sum_{k=1}^{D} \frac{|S_k|}{k^{2-\gamma}}.$$

Let λ be taken so large that

$$\left|\sum_{k=\lambda D+1}^{\infty} \frac{\chi_D(k)}{k^s}\right| < 1.$$

In this case

$$|L(s,\chi)| < c_5 \sum_{k=1}^{D} \frac{|S_k|}{k^{2-\gamma}}.$$

But

$$\sum_{k=1}^{D}\frac{|S_k|}{k^{2-\gamma}} \leq \sum_{k=1}^{D^\omega}\frac{1}{k^{1-\gamma}} + \prod_{i=1}^{l} p_i^{-\beta_i}\sum_{k>D^\omega}^{D}\frac{1}{k^{1-\gamma}} \leq c_6\frac{D^{\omega\gamma}+D^\gamma\prod_{i=1}^{l}p_i^{-\beta_i}}{\gamma}.$$

Consequently,

$$|L(s,\chi)| < c_7\frac{D^{\omega\gamma}+D^\gamma\prod_{i=1}^{l}p_i^{-\beta_i}}{\gamma},$$

and, taking into account the conditions of the lemma, we obtain

$$|L(s,\chi)| < c_2/\gamma.$$

Proof of the theorem. For the proof it is enough to show that, if

$$\gamma = (\log D)^{-\frac{Q}{Q+1}}, \qquad \log D \leq \left(\alpha_\nu/\log^3\alpha_\nu\right)^{\frac{Q+1}{4}},$$

then the conditions of the lemma are satisfied, i.e. $\omega\gamma \leq c_0/\log D$ and $\alpha_i\gamma \leq \beta_i$, $i = 1,\dots,l$. But, applying Theorem 1, we have $\omega = 3(\log^3\alpha_\nu/\alpha_\nu)^{1/4}$, $\beta_i = \frac{\alpha_i}{\alpha_\nu}(\alpha_\nu\log\alpha_\nu)^{1/4}$.

The conditions of the lemma take the form:

a) $3(\log^3\alpha_\nu/\alpha_\nu)^{1/4}(\log D)^{Q+1} \leq c_0$;

b) $\alpha_\nu^{3/4}/(\log\alpha_\nu)^{1/4} < (\log D)^{\frac{Q}{Q+1}}$.

Then a) follows from the fact that $\log D \leq (\alpha_\nu/\log^3\alpha_\nu)^{1/4}$, while b) is obvious for $Q > 3$. Thus the theorem is proved.

Theorem 3. If $\chi_D(k)$ is a primitive character modulo D and

$$\log D \leq \left(\alpha_\nu/\log^3\alpha_\nu\right)^{\frac{Q+1}{4}}, \qquad Q > 3 \text{ is a constant}, \qquad (\alpha_\nu/\log^3\alpha_\nu)^{1/4} \geq 13,$$

then $L(s,\chi)$ has no zeros in the region

$$\sigma > 1 - \frac{c}{\log^{\frac{Q}{Q+1}} D\cdot\log\log D}.$$

The proof is not given here, since it is entirely analogous to the proof of the theorem for $D = p^n$ (see (1), Theorem 3).

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Note: Figure translations are in progress. See original paper for figures.