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Abstract

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THE E-GREEN FUNCTION OF THE BELTRAMI OPERATOR AND SOME VARIATIONAL PROBLEMS

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1. From the results of M. V. Keldysh ([1]) it follows that for the equation

$$B_k(u) = \frac{\partial}{\partial x}\left\{y^k \frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^k \frac{\partial u}{\partial y}\right\} = 0 \qquad (1)$$

for $k > 1$ the so-called problem E is uniquely solvable.

Problem E. Find a twice continuously differentiable and bounded solution of equation (1) in the domain Ω^+, adjacent to the axis Ox, which assumes the prescribed continuous values $F(M)$ on the part Γ^+ of the boundary $\partial\Omega^+$ situated in the upper half-plane.

The solution of problem E in the case of the semicircle $(K^+\{x^2 + y^2 < R^2,\ y > 0\})$ can be obtained in closed form with the aid of the fundamental solution $G(x, y; \xi, \eta)$ of equation (1), possessing the properties:

a) $G(x, y; \xi, \eta) = G(\xi, \eta; x, y)$;

b) $G(x, y; \xi, \eta) = 0$ on the semicircle $\Gamma^+\{x^2 + y^2 = R^2,\ y > 0\}$;

c) $G\big|_{y=0}$ is bounded; $\partial G/\partial y\big|_{y=0} = 0$.

We call this solution the E-Green function for the semicircle K^+. For the construction of the function G we use the fundamental solution of equation (1), found by A. Weinstein ([2]):

$$g(x, y; \xi, \eta) = \int_0^\pi \frac{\sin^{k-1}\alpha\, d\alpha}{[(x-\xi)^2 + \eta^2 + y^2 - 2y\eta\cos\alpha]^{k/2}}.$$

Guided by the method of electrostatic images, we obtain G in the form

$$G(x, y; \xi, \eta) = \frac{1}{2\pi}\left\{-g(x, y; \xi, \eta) + \left(\frac{R}{\rho}\right)^k g(x, y; \xi^*, \eta^*)\right\},$$

where $\rho = \sqrt{\xi^2 + \eta^2}$, and (ξ^*, η^*) is the point conjugate to (ξ, η) with respect to the circle $x^2 + y^2 = R^2$.

Using now Green' s formula

$$\int_{K^+} \{uB_k(v) - vB_k(u)\}\,dx\,dy = \int_{\partial K^+} y^k \left\{u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right\} ds,$$

we arrive in the usual way at the representation of the solution

$$u_R(\xi,\eta) = \int_{\Gamma^+} F(M)\, y^k \frac{\partial G}{\partial n}\, ds_M,$$

which, after appropriate calculations, takes the form

$$u_R(\xi,\eta) = \frac{kR^k}{2\pi}\int_0^\pi \sin^k\varphi\cdot F(\varphi)\left\{\int_0^\pi \frac{(R^2-\xi^2-\eta^2)\sin^{k-1}\alpha\,d\alpha}{\left[(R\cos\varphi-\xi)^2+R^2\sin^2\varphi+\eta^2-2R\eta\cos\alpha\cdot\sin\alpha\right]^{(k+2)/2}}\right\} d\varphi. \tag{2}$$

By direct verification one can see that formula (2) gives a solution of problem E. It is important for us to note that this solution satisfies the condition $\partial u_R(x,y)/\partial y\big|_{y=0} = 0$, which follows at once from the analyticity of the function $u_R(x,y)$ in a neighborhood of any point of the segment $\{-R < x < R,\ y = 0\}$ and from its evenness with respect to y. Moreover, formula (2) gives an analytic solution of equation (1) in the whole disk $x^2 + y^2 < R^2$.

Remark 1. Formula (2) can without difficulty be extended to the n-dimensional case. In this case it gives a solution of problem E for all $k > 0$. However, when the parameter k varies in the interval $(0,1)$, this solution is not unique. We refer to paper ([3]), where, with the aid of Green' s function constructed on the basis of another fundamental solution, a solution of the Dirichlet problem is written down for the n-dimensional equation (1) with $k < 1$ in the half-ball $^+:\ x_1^2 + \cdots + x_n^2 < R^2,\ x_n > 0$.

Remark 2. It is also important to note (we shall use this fact later) that formula (2) gives a solution of equation (1) under considerably more general requirements on the boundary function. In this case it may also represent unbounded solutions. Naturally, in these cases questions arise as to how the fulfillment of the boundary conditions on Γ^+ is to be understood and as to uniqueness of the solution.

Remark 3. With the aid of the fundamental solution

$$-\frac{1}{2\pi} g(x,y;\xi,\eta)$$

one can construct, by the usual methods, the E-Green' s function for other domains as well; for example, the method of electrostatic images is suitable for segments of a disk, for the first quadrant, etc.

2. We now pass to the variational analogue of problem E. We shall consider a simply connected domain Ω^+ with piecewise smooth boundary Γ^*, adjacent from above to the axis Ox. By the trace of a function $u(x,y)$, defined in Ω^+, on a smooth piece $\gamma \subset \Gamma$ we shall mean a function $F(M)$ for which

$$\lim_{\substack{a\\(x,y)\to M}} u(x,y) = F(M)$$

for almost all points $M \in \gamma$ (the limit is taken along any continuously differentiable field of directions $a(x,y)$ not tangent to γ, and $F(M)$, up to equivalence, does not depend on the particular choice of a.

Problem E_{var}**.** Find an (analytic) solution $u(x,y)$ of equation (1) having on Γ^+ the prescribed trace $F(M)$ and possessing the finite weighted integral

$$D_k(u) = \iint_{\Omega^+} y^k \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} dx\, dy, \qquad k > 1. \tag{3}$$

The study of this problem is natural, since the operator $B_k(u)$ is the gradient of the functional (3), and consequently the variational method can be used in its solution. This was done in the work of L. D. Kudryavtsev ([4]), where it is proved that there exists a unique solution of the variational problem for the minimum of the functional (3), and that this solution is an analytic function which represents the unique, under the condition $D_{k-\varepsilon}(u) < \infty$, solution of equation (1). We supplement the result of L. D. Kudryavtsev in several respects. First, we give necessary and at the same time sufficient conditions for the solvability of problem E_{var} in terms of the properties

* The boundary Γ is divided into a segment Γ_0 lying on the axis Ox and a segment Γ^+ lying in the upper hal

boundary function $F(M)$. Secondly, we refine the uniqueness theorem. Finally, we prove that on the free part of the boundary Γ_0 the "natural" boundary conditions $\partial u/\partial n\big|_{\Gamma_0} = 0$ are satisfied, which leads to an improvement of some estimates and has interesting consequences.

Theorem 1. *In order that the problem* E_{var} *have a solution, it is necessary and sufficient that the conditions*

$$\begin{aligned} &1)\quad \int_{\Gamma^+} y^k |F(M)|^2\, ds < \infty; \\ &2)\quad \int_{\Gamma^+} ds_M \int_{\Gamma^+} \frac{|F(M)-F(Q)|^2}{|MQ|^2}\, \omega^k(M,Q)\, ds_Q < \infty, \end{aligned} \tag{4}$$

be satisfied, where $\omega(M,Q)$ is the smaller of the distances of the points M and Q to the axis Ox. *

The proof proceeds by reducing the problem to a variational problem according to the schemes used in (4). The fact that conditions 1) and 2) constitute a criterion for the boundedness of $D_k(u)$ is established with the aid of the apparatus described in (5).

Theorem 2. *The solution of the problem E_{var} is unique.*

We first establish an auxiliary lemma. Denote by $W^{(1)}_{2,k}(\Omega^+)$ the class of square-summable functions in Ω^+ that possess generalized derivatives, in the sense of S. L. Sobolev, with finite integral $D_k(u)$.

Lemma. *Let the function $\vartheta \in W^{(1)}_{2,k}(\Omega^+)$ possess second generalized derivatives and almost everywhere satisfy the equation $B_k(\vartheta) = 0$. Then for any function $\zeta(x,y) \in W^{(1)}_{2,k}$ that vanishes on Γ^+, the equality (the equation in variations)*

$$\iint_{\Omega^+} y^k \left(\frac{\partial \vartheta}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \vartheta}{\partial y}\frac{\partial \zeta}{\partial y} \right) dx\, dy = 0$$

holds.

Proof. Let Ω_δ be the set of points of the domain Ω^+ whose distance to the axis Ox is greater than δ, and let I_δ be the intersection of the domain Ω^+ with the straight line $y = \delta$. By integration by parts and with the aid of Bunyakovsky' s inequality, we obtain the estimate

$$\left\{ \iint_{\Omega_\delta} y^k \left(\frac{\partial \vartheta}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \vartheta}{\partial y}\frac{\partial \zeta}{\partial y} \right) dx\, dy \right\}^2 = \left(\int_{I_\delta} y^k \zeta \frac{\partial \vartheta}{\partial y}\, dx \right)^2 \le y^2 \int_{I_\delta} y^k \left(\frac{\partial \vartheta}{\partial y} \right)^2 dx \cdot \int_{I_\delta} y^{k-2} \zeta^2 dx = \delta^2 J(\delta) \cdot L(\delta),$$

$$J(\delta) = \int_{I_\delta} \delta^2 \left(\frac{\partial \vartheta}{\partial y} \right)^2 \Big|_{y=\delta} dx, \qquad L(\delta) = \int_{I_\delta} \delta^{k-2} \zeta^2(x,\delta)\, dx.$$

From the finiteness of $D_k(u)$ there follows the existence of the integral $\int_0^H J(\delta)\, d\delta$, where H is the vertical size of the domain Ω. Moreover, from Hardy' s inequality (4) there follows the estimate

$$\iint_{\Omega^+} y^{k-2} \zeta^2(x,y)\, dx\, dy \le C \iint_{\Omega^+} y^k \left(\frac{\partial \zeta}{\partial y} \right)^2 dx\, dy,$$

and hence for almost all δ the integral $L(\delta)$ also exists. But then, at least for some sequence δ_k, the estimate must hold:

$$J(\delta_k) \le \frac{c}{\delta_k |\ln \delta_k|}, \qquad L(\delta_k) \le \frac{c}{\delta_k |\ln \delta_k|},$$

* Under our conditions, the trace $F(M)$ is simultaneously a boundary value in the sense of convergence in the quadratic mean on any interior part of Γ^+, and, after adding the weight y^k, also on all of Γ^+.

and, consequently, on passing to the limit along this sequence we obtain

$$\lim_{\delta_k \to 0} \iint_{\delta\Omega_k} y^k \left(\frac{\partial \vartheta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial \vartheta}{\partial y}\frac{\partial \xi}{\partial y} \right) dx\, dy = 0.$$

Since the integral exists (which follows from Bunyakovsky' s inequality), it follows, by the property of absolute continuity of the Lebesgue integral, that the assertion of the lemma follows.

The proof of Theorem 2 now proceeds as in (6) (p. 105).

3. Let us now prove that the solution u of the problem E_{var} satisfies (in the classical sense) the condition $\partial u/\partial y|_{y=0} = 0$. Consider an interior point M of the segment Γ_0, and let K_R^+ be a semicircle in the upper half-plane of sufficiently small radius R with center at the point M. The function u has on the upper boundary Γ_R^+ of this circle the value $F_R(\varphi)$, which, by Theorem 1, satisfies conditions (4). Construct, by formula (2), the function $u_R(x, y)$. By direct calculation one can verify the validity of the inequality (ensuring the finiteness of $D_k(u_R, K_R^+)$)

$$D_k(u_R, K_R^+) \le c \int_{\Gamma_R^+} ds_M \int_{\Gamma_R^+} \frac{|F_R(M) - F_R(Q)|^2}{|MQ|^2}\, \omega^k(M, Q)\, ds_Q < \infty.$$

Since $u_R(x, y)$ is a solution of equation (1) and has a bounded weighted integral, then, in view of the results mentioned in work (4), it is at the same time the unique solution of the variational problem in the semicircle K_R^+. But then it must coincide with $u(x, y)$, for otherwise the function

$$\tilde{u}(x, y) = \begin{cases} u(x, y), & \text{in } \Omega^+ - K_R^+, \\ u_R(x, y), & \text{in } K_R^+, \end{cases}$$

would have a smaller weighted integral than u, which contradicts the minimal property of u (which is established in the course of the proof of Theorem 1). Thus, $u_R \equiv u$, and therefore $\partial u/\partial n|_{y=0} = 0$. On the basis of the arguments just given, one can assert even more.

Corollary. The solution of the problem $\dot{E}_{\text{var}}$, when continued evenly with respect to y, gives an analytic solution of equation (1) in the domain $\Omega^+ + \Gamma_0 + \Omega^- *$.

The analogous assertion is, of course, also valid for problem E. We note that the solution of the problem E_{var} need not be bounded. It may have isolated discontinuities of logarithmic type on Γ^+ and discontinuities of power type at the endpoints of Γ_0.

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* Ω^- is the mirror reflection of the domain Ω^+ in the axis Ox.

Note: Figure translations are in progress. See original paper for figures.