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1961

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Abstract

Full Text

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ON THE REGULARIZATION PROBLEM IN THE THEORY OF INTEGRAL EQUATIONS WITH CAUCHY KERNEL

(Presented by Academician N. I. Muskhelishvili, 22 IV 1961)

Let us consider the integral equation with Cauchy kernel

$$N\varphi \equiv a(t)\varphi(t) + b(t)S\varphi + V\varphi = f(t), \tag{1}$$

where

$$S\varphi \equiv \frac{1}{\pi i}\int_\Gamma \frac{\varphi(t_1)}{t_1 - t}\,dt_1, \qquad t \in \Gamma,$$

and Γ is a finite collection of mutually nonintersecting, simple, closed and open Lyapunov lines; $a(t)$, $b(t)$, $f(t)$ are functions prescribed on Γ; $\varphi(t)$ is the unknown function; V is a completely continuous operator in the space in which $\varphi(t)$ is sought, and the integral is understood in the sense of the Cauchy principal value.

If $a(t)$, $b(t)$, V are square matrices, and $\varphi(t)$, $f(t)$ are vectors, then (1) represents a system of integral equations with Cauchy kernel.

One of the principal problems considered in connection with the study of equation (1) is the problem of regularization. This problem consists in the following: it is required to find an operator M of the form N such that the composition MN have the form $MN\varphi = \varphi + T\varphi$, where T is a completely continuous operator. In this case the operator M is called a left regularizer of equation (1), or of the operator N. A right regularizer is defined analogously.

The solution of the regularization problem has a twofold significance: first, the solution of this problem makes it possible to reduce the determination of solutions of non-Fredholm equations to the determination of solutions of Fredholm equations; second, it is an important step in establishing general properties of integral equations with Cauchy kernel, in particular Noether theorems.

If we assume that Γ consists only of closed lines, that the functions $a(t)$, $b(t)$ satisfy the Hölder condition, and that $a^2(t) - b^2(t) \neq 0$ everywhere on Γ, then the regularization problem is solved simply. Both a left and a right regularizer in this case will be, as is known, for example, the operator

$$M\varphi \equiv a_0(t)\varphi(t) - S(b_0\varphi), \tag{2}$$

where

$$a_0(t) = \frac{a(t)}{a^2(t) - b^2(t)}, \qquad b_0(t) = \frac{b(t)}{a^2(t) - b^2(t)}.$$

Indeed, simple computations show that

$$MN\varphi \equiv \varphi + T\varphi, \qquad NM\varphi \equiv \varphi + \widehat{T}\varphi,$$

where

$$T = -\frac{1}{\pi i}T_{ab_0} - \frac{1}{\pi i}T_{bb_0}S + MV,$$

$$\widehat{T} = -\frac{a}{\pi i}T_{b_0} + \frac{b}{\pi i}T_{a_0} + VM, \tag{3}$$

and the operator T_ω is defined by the formula

$$T_\omega\varphi \equiv \int_\Gamma \frac{\omega(t_1) - \omega(t)}{t_1 - t}\varphi(t_1)\, dt_1.$$

In cases where open lines also participate in Γ, or the coefficients of the equation $a(t), b(t)$ have discontinuities of the first kind, the regularization problem has been solved ([1,2]) only by invoking the Riemann problem (meaning the boundary-value problem $\Phi^+(t) = G(t)\Phi^-(t) + g(t)$). As is known, this problem for one unknown piecewise-holomorphic function is solved effectively in quadratures, whereas for several unknown piecewise-holomorphic functions, in the general case, it cannot be solved effectively. Therefore, when the Riemann problem is invoked in solving the regularization problem, then in the case of a system of integral equations with Cauchy kernel one can show only that the regularization problem has a solution, but we have no possibility of effectively constructing the corresponding system of Fredholm integral equations. Moreover, regularization methods based on the Riemann problem plainly exclude the possibility of applying them to the case of multiple singular integral equations.

In the paper ([3]) the regularization problem in the cases where the line Γ is open, or where the coefficients $a(t), b(t)$ have discontinuities of the first kind, was solved without invoking the Riemann problem, by constructing special regularizers distinct from those used in the case of closed Γ and continuous coefficients. In the present note we show that the operator which regularizes an integral equation with Cauchy kernel in the case of continuous coefficients and closed lines also regularizes the equation in the case where the lines are open and the coefficients are discontinuous.

When the line Γ is closed and the coefficients $a(t), b(t)$ are continuous in the Hölder sense (with $a^2 - b^2 \neq 0$), regularization of equation (1), as indicated above, is performed, for example, by the operator M. In this case the proof that equations (3), obtained after regularization, are Fredholm equations reduces to proving the complete continuity of the operator T_ω, and the function $\omega(t)$ must be subjected to the same restrictions as we impose on the coefficients of the equation $a(t), b(t)$.

In the paper ([4]) it is shown that if the function $\omega(t)$ is continuous on Γ, then the operator T_ω is completely continuous in the Hilbert functional space $L_2(\Gamma)$. This result enabled S. G. Mikhlin to solve the regularization problem in the case where Γ is closed and the coefficients $a(t), b(t)$ are continuous.

We now generalize the result established in ([4]) concerning the operator T_ω. To formulate the new result, we introduce some generalizations.

If the function $\varphi(t)$ has discontinuities of the first kind at the points $c_1, c_2, \dots, c_m$ of the line Γ, and on each closed part of this line lying between neighboring points c_k it is continuous, then we shall say that $\varphi(t)$ belongs to the class $C(\Gamma; c_1, c_2, \dots, c_m)$.

Let α_k be real numbers such that $0 \leq \alpha_k < 1$, $k = 1, \dots, m$. Consider the function

$$\rho(t) = \prod_{k=1}^{m_1} |t - c_k|^{\alpha_k(p-1)} \prod_{k=m_1+1}^{m} |t - c_k|^{-\alpha_k}, \qquad 0 \leq m_1 \leq m;$$

when $m_1 = 0$ we set the first product equal to unity, and when $m_1 = m$ we set the second product equal to unity. By $L_p(\Gamma; \rho)$ we shall denote the functional space whose elements are functions defined on Γ and integrable in the p-th power with weight $\rho(t)$. The norm of an element $\varphi \in L_p(\Gamma; \rho)$ is defined by the formula

$$\|\varphi\|^p = \int_\Gamma \rho(t)\, |\varphi(t)|^p \, ds,$$

and we shall always suppose that $p > 1$.

Theorem 1. *If $\omega(t) \in C(\Gamma; c_1, c_2, \dots, c_m)$, then the operator T_ω maps the space $L_p(\Gamma; \rho)$ into itself and is completely continuous.*

The fact that the operator T_ω maps the space $L_p(\Gamma; \rho)$ into itself follows from the fact that the operator S has this property ([3]).

Let us now consider the function

$$\omega_1(t) = \prod_{k=1}^{m_1} |t - c_k|^{\alpha_k} \prod_{k=m_1+1}^{m} |t - c_k|^{\beta_k}, \qquad 0 < \beta_k < \frac{1-\alpha_k}{p},$$

$$k = m_1 + 1, \dots, m.$$

The multiplication operator $B\varphi = \omega_1(t)\varphi$ maps the space $L_p(\Gamma; \rho)$ isometrically onto the space $L_p(\Gamma; \rho_1)$, where

$$\rho_1(t) = \prod_{k=1}^{m_1} |t - c_k|^{1-\alpha_k} \prod_{k=m_1+1}^{m} |t - c_k|^{1-(\alpha_k + p\beta_k)}.$$

It is easy to verify that the identity

$$\omega_1(t) T_\omega \varphi = T_{\omega\omega_1}\varphi - T_{\omega_1}(\omega\varphi)$$

holds.

The operators appearing in this equality map the space $L_p(\Gamma; \rho)$ into the space $L_p(\Gamma; \rho_1)$. Moreover, since the functions $\omega_1(t)$, $\omega(t)\omega_1(t)$ are continuous on Γ, the operators $T_{\omega\omega_1}\varphi$, $T_{\omega_1}(\omega\varphi)$ are completely continuous ([3]). Finally, from the obvious equality $T_\omega = B^{-1} B T_\omega$ it follows directly that the operator T_ω is completely continuous.

In an analogous way one can consider a number of cases in which the function $\omega(t)$ has infinite discontinuities at a finite number of points of the line Γ. Thus, for example, the following holds:

Theorem 2. If $\sigma(t)\omega(t) \in C(\Gamma; c_1, c_2, \dots, c_m)$, where

$$\sigma(t) = \prod_{k=1}^{m} |t - c_k|^{\alpha_k}, \qquad 0 < \alpha_k < 1,$$

then the operator T_ω maps the space $L_p(\Gamma; \sigma^{-1})$ into the space $L_p(\Gamma; \delta^{p-1})$ and is completely continuous.

The proof follows easily from the preceding theorem and the identity

$$\sigma T_\omega \varphi = T_{\sigma\omega}\varphi - T_\sigma(\omega\varphi).$$

Suppose now that the coefficients of equation (1), $a(t)$ and $b(t)$, belong to $C(\Gamma; c_1, c_2, \dots, c_m)$; $a^2(t) - b^2(t) \neq 0$ everywhere on Γ; $f(t)$, $\varphi(t) \in L_p(\Gamma; \rho)$; V maps $L_p(\Gamma; \rho)$ into itself and is completely continuous; and Γ consists of closed lines (since the coefficients of the equation may have discontinuities of the first kind, the case of an open line can be reduced to the case of a closed one ([3])).

Taking into account Theorem 1 and equalities (3), we conclude that, under the above assumptions, the operator M is both a left and a right regularizer of the operator N in the space $L_p(\Gamma;\rho)$. Hence, in turn, from the corresponding result of F. V. Atkinson ([5]) it follows directly that, for equation (1), the Noether theorems hold in the space $L_p(\Gamma;\rho)$.

One may also consider cases in which the coefficients of equation (1) are unbounded. The regularization problem for a system of equations of the form (1) is solved analogously.

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Received
7 IV 1961

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Note: Figure translations are in progress. See original paper for figures.