ALGORITHMS FOR SIMPLIFYING DISJUNCTIVE NORMAL FORMS OF FINITE INDEX

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Abstract

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ALGORITHMS FOR SIMPLIFYING DISJUNCTIVE NORMAL FORMS OF FINITE INDEX

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In the present note we use the concepts and notation introduced in ([2],[3]). Instead of the notation $\varphi[S(\mathfrak{A}^{(i)}, \mathfrak{N})]$ we introduce the more natural notation $\varphi[\mathfrak{A}^{(i)}, S(\mathfrak{A}^{(i)}, \mathfrak{N})]$.

It is known ([3]) that in the class of algorithms of finite index there is no algorithm which, for all functions f of the algebra of logic, from a reduced DNF $\mathfrak{N}_f$ makes it possible to construct a DNF $(\mathfrak{N}_f)_{\Sigma M}$. At the same time, algorithms of finite index make it possible in a number of cases to carry out considerable simplifications of a reduced DNF. In the present note we study algorithms of finite index that make it possible to carry out the greatest possible simplifications of a DNF.

Application of such algorithms to a reduced DNF $\mathfrak{N}_f$ makes it possible to construct DNFs

$$(\mathfrak{N}_f)_1, \dots, (\mathfrak{N}_f)_k, \dots \supseteq (\mathfrak{N}_f)_1 \supseteq \cdots \supseteq (\mathfrak{N}_f)_k \supseteq \cdots,$$

possessing the principal properties of a reduced DNF:

1) to each function f of the algebra of logic there corresponds a unique DNF $(\mathfrak{N}_f)_i,\ i = 1, 2, \dots, k, \dots;$
2) $(\mathfrak{N}_f)_{\Sigma M} \subseteq (\mathfrak{N}_f)_i,\ i = 1, 2, \dots, k, \dots;$
3) it also turns out that, by applying algorithms of index k to the reduced DNF $\mathfrak{N}_f$, one cannot obtain a DNF simpler than $(\mathfrak{N}_f)_k$.

Let $\mathfrak{N}_f$ be a reduced DNF of the function f, composed of unmarked conjunctions; $\mathfrak{N}_f(A)$, $\mathfrak{N}_f(B)$ are the images of $\mathfrak{N}_f$ under algorithms A and B, respectively. We shall say that the inequality $A \geqslant B$ holds if, for all f, the relation $\mathfrak{N}_f(A) \supseteq \mathfrak{N}_f(B)$ is fulfilled.

An algorithm A of index k is called k-majorant if, for every algorithm B of index k, the relation $A \geqslant B$ holds.

If all algorithms generated by a function φ are k-majorant, then the function φ will also be called k-majorant.

We note that if an algorithm A is generated by a monotone function φ and is k-majorant, then the function φ is k-majorant.

We proceed to the construction of k-majorant functions $\varphi_k[\mathfrak{A}^{(i)}, S(\mathfrak{A}^{(i)}, \mathfrak{N})]$ ($k = 1, 2, \dots, m, \dots$). In the constructions we use functions $f(x_1 \dots x_n)$, defined on a nonempty subset of the set E_n of all vertices of the n-dimensional unit cube and taking the values 0 and 1 ([1]) (not everywhere defined functions of the algebra of logic).

Let a nonempty set $\mathfrak{M}$ of functions of the algebra of logic and not everywhere defined functions of the algebra of logic be given. A conjunction $\mathfrak{A}$ has property (0) relative to $\mathfrak{M}$ if in $\mathfrak{M}$ there exist functions f_1 and f_2 such that

$$\mathfrak{A} \in (\mathfrak{N}_{f_1})_{\Sigma M}, \qquad \mathfrak{A} \notin (\mathfrak{N}_{f_2})_{\Sigma M}.$$

A conjunction $\mathfrak{A}$ has property (1) relative to $\mathfrak{M}$ if: 1) for all f from $\mathfrak{M}$ we have

$$\mathfrak{A} \in (\mathfrak{N}_f)_{\Sigma M};$$

2) in $\mathfrak{M}$ there exist functions f_1 and f_2 such that

$$\mathfrak{A} \in (\mathfrak{N}_{f_1})_{\cap M}, \qquad \mathfrak{A} \notin (\mathfrak{N}_{f_2})_{\cap M}.$$

A conjunction $\mathfrak{A}$ has property (2) relative to $\mathfrak{M}$ if, for all f from $\mathfrak{M}$, we have

$$\mathfrak{A} \notin (\mathfrak{N}_f)_{\Sigma M}.$$

A conjunction $\mathfrak{A}$ has ⋯

has property (3) with respect to $\mathfrak{N}$, if for all f from $\mathfrak{N}$ the relations $\mathfrak{A} \in (\mathfrak{N}_f)_{\Sigma M}$, $\mathfrak{A} \notin (\mathfrak{N}_f)_{\cap M}$ are satisfied. The conjunction $\mathfrak{A}^{(i)}$ has property (4) with respect to $\mathfrak{N}$, if for all f from $\mathfrak{N}$ the relation $\mathfrak{A} \in (\mathfrak{N}_f)_{\cap M}$ is satisfied.

Definition. We shall call a conjunction $\mathfrak{A}^{(f)}$, $j \in \{0, 1, 2, 3, 4\}$, and a function f **consistent** if one of the conditions 1)–5) is fulfilled: 1) $j = 0$; 2) $j = 1$, $\mathfrak{A} \in (\mathfrak{N}_f)_{\Sigma M}$; 3) $j = 2$, $\mathfrak{A} \notin (\mathfrak{N}_f)_{\Sigma M}$; 4) $j = 3$, $\mathfrak{A} \in (\mathfrak{N}_f)_{\Sigma M}$, $\mathfrak{A} \notin (\mathfrak{N}_f)_{\cap M}$; 5) $j = 4$, $\mathfrak{A} \in (\mathfrak{N}_f)_{\cap M}$. A set of conjunctions $\{\mathfrak{A}_1^{(j_1)}, \mathfrak{A}_2^{(j_2)}, \dots, \mathfrak{A}_l^{(j_l)}\}$ and f are consistent if all $\mathfrak{A}_i^{(j_i)}$ ($i = 1, 2, \dots, l$) are consistent.

Let $\mathfrak{N}_f$ be a reduced d.n.f. of a Boolean-algebra function $f(x_1, \dots, \dots, x_n)$, $\mathfrak{A}^{(j)} \in \mathfrak{N}_f$, and

$$S_k(\mathfrak{A}^{(j)}, \mathfrak{N}_f) = \{\mathfrak{A}^{(j)}, \mathfrak{B}_1^{(j)}, \dots, \mathfrak{B}_l^{(j)}\}$$

be the principal neighborhood of order k of the conjunction $\mathfrak{A}^{(j)}$ in the d.n.f. $\mathfrak{N}_f$ ($S_0(\mathfrak{A}^{(j)}, \mathfrak{N}_f) = \{\mathfrak{A}^{(j)}\}$).

Denote by $K_n(\mathfrak{A}^{(j)}, S_k, \mathfrak{N}_f)$ the set of Boolean-algebra functions $\psi_i(x_1, \dots, x_n)$ such that: 1) $M[S_k(\mathfrak{A}^{(j)}, \mathfrak{N}_{\psi_i})] = M[S_k(\mathfrak{A}^{(j)}, \mathfrak{N}_f)]$; 2) ψ_i and $S_k(\mathfrak{A}^{(j)}, \mathfrak{N}_f)$ are consistent. The conjunction $\mathfrak{A}^{(j)}$ has, with respect to $K_n(\mathfrak{A}^{(j)}, S_k, \mathfrak{N}_f)$, one of the properties (0), (1), (2), (3), (4).

We define the functions $\varphi_k(\mathfrak{A}^{(j)}, S_k(\mathfrak{A}^{(j)}, \mathfrak{N}_f))$, $k = 1, 2, \dots, m, \dots$, as follows:

$\mathfrak{A}^{(j)}$ has, with respect to $K_n(\mathfrak{A}^{(j)}, S_k, \mathfrak{N}_f)$, property (i), $i \in \{0, 1, 2, 3, 4\}$;

$$\varphi_k[\mathfrak{A}^{(j)}, S_k(\mathfrak{A}^{(j)}, \mathfrak{N}_f)] = (i). \tag{1}$$

Theorem 1. The functions $\varphi_k[\mathfrak{A}^{(j)}, S_k(\mathfrak{A}^{(j)}, \mathfrak{N}_f)]$ are monotone ($k = 1, 2, \dots, \dots, m, \dots$).

It follows from Theorem 1 that all algorithms generated by the function φ_k are equivalent. We shall therefore speak of a single algorithm generated by the function φ_k, and denote it by A_k.

Theorem 2. The functions φ_k and, consequently, the algorithms A_k are k-majorant.

Applying to $\mathfrak{N}_f$ the algorithms generated by the functions $\varphi_1, \varphi_2, \dots, \varphi_m, \dots$, we obtain d.n.f.' s $(\mathfrak{N}_f)_1, \dots, (\mathfrak{N}_f)_m, \dots$, possessing properties (1), (2), (3).

Let us also note that, for any $m > 0$, there is a function $f(x_1, \dots, \dots, x_{n(m)})$ such that $M[(\mathfrak{N}_f)_m \setminus (\mathfrak{A}_f)_{m-1}]$ is nonempty.

The computation of φ_k, based on definition (1), is connected with great difficulties. Below we shall construct functions Φ_k, $k = 1, 2, \dots, m, \dots$, which generate algorithms weaker than φ_k, but more simply computable.

Let $\mathfrak{A}^{(j)} \in \mathfrak{N}_f$ and

$$S_k(\mathfrak{A}^{(j)}, \mathfrak{N}_f) = \{\mathfrak{A}^{(j)}, \mathfrak{B}_2^{(i_1)}, \dots, \mathfrak{B}_l^{(i_l)}\}, \qquad S_{k-1}(\mathfrak{A}^{(j)}, \mathfrak{N}_f) = \{\mathfrak{A}^{(j)}, \mathfrak{A}_1^{(i_1)}, \dots, \mathfrak{A}_p^{(i_p)}\}.$$

We denote:

$$M_k(\mathfrak{A}^{(j)}, \mathfrak{N}_f) = \bigcup_{i=1}^{l} N_{\mathfrak{B}_i} \cup N_{\mathfrak{A}},$$

$$M_{k-1}(\mathfrak{A}^{(j)}, \mathfrak{N}_f) = \bigcup_{i=1}^{p} N_{\mathfrak{A}_i} \cup N_{\mathfrak{A}}, \qquad M_{k-1,k} = M_k \setminus M_{k-1}.$$

We shall denote the set of all subsets of $M_{k-1,k}$ by $\overline{M}_{k-1,k}$; the set of all vertices of the n-dimensional unit cube by $\overline{E}_n$. To each element $\overline{m}_\alpha$ of $\overline{M}_{k-1,k}$ we assign a not everywhere defined Boolean-algebra function-

vertices of logic:

$$f(x_1 \dots x_n) = \begin{cases} 1, & \text{if } (x_1 \dots x_n) \in [M_k(\mathfrak{A}^{(j)}, \mathfrak{A}_l) \setminus \overline{\mathfrak{M}}_a], \\ 0, & \text{if } (x_1 \dots x_n) \in [E_n \setminus M_k(\mathfrak{A}^{(j)}, \mathfrak{A}_l)], \\ \text{undefined}, & \text{if } (x_1 \dots x_n) \in \overline{\mathfrak{M}}_a. \end{cases}$$

We shall denote the set of functions thus obtained by $\{F_{M_k}\}$. From $\{F_{M_k}\}$ we single out all functions with which the conjunctions $\mathfrak{A}^{(j)}, \mathfrak{B}_1^{(i_1)}, \dots, \mathfrak{B}_l^{(i_l)}$ are compatible. We shall denote the set of such functions by $\{F_{SM_k}\}$.

Define the function $\widetilde{\varphi}_k$ as follows:
$\mathfrak{A}^{(j)}$ has, relative to $\{F_{SM_k}\}$, property (i) $(i \in \{0, 1, 2, 3, 4\})$.

$$\widetilde{\varphi}_k[\mathfrak{A}^{(j)}, S_k(\mathfrak{A}^{(j)}, \mathfrak{A}_l)] = (i). \tag{11}$$

Theorem 3. The functions $\widetilde{\varphi}_k$, $k = 1, 2, \dots, m, \dots$, are monotone.

Using definition (11), it is not difficult to construct an algorithm for computing $\widetilde{\varphi}_k$.

In the set M_k select all points at distance not greater than 1* from $M_{k-1,k}$. Denote the resulting set by $M^0_{k-1,k} \cup M^1_{k-1,k}$.

Theorem 4. If the neighborhood $S_k(\mathfrak{A}^{(j)}, \mathfrak{A}(x_1 \dots x_n))$ is such that the set $M^0_{k-1,k} \cup M^1_{k-1,k}$ is contained in the set of vertices of a face of dimension m, and $m \leq n - 2$, then

$$\varphi_k[\mathfrak{A}^{(j)}, S_k(\mathfrak{A}^{(j)}, \mathfrak{A}(x_1 \dots x_n))] = \widetilde{\varphi}_k[\mathfrak{A}^{(j)}, S_k(\mathfrak{A}^{(j)}, \mathfrak{A}(x_1 \dots x_n))].$$

We shall call the neighborhood $S_k(\mathfrak{A}^{(j)}, \mathfrak{A}(x_1 \dots x_n))$ **nondegenerate** for $(\mathfrak{A}^{(j)}, \mathfrak{A})$ if in $S_k(\mathfrak{A}^{(j)}, \mathfrak{A})$ there is no conjunction $\mathfrak{B}^{(i)}$ such that $N_{\mathfrak{B}} \subseteq M_{k-1,k}$.

Theorem 5. If $S_k(\mathfrak{A}^{(j)}, \mathfrak{A})$ is nondegenerate for $(\mathfrak{A}^{(j)}, \mathfrak{A}(x_1 \dots x_n))$, and the set $M^0_{k-1,k} \cup M^1_{k-1,k}$ is contained in the set of vertices of a face of dimension not greater than $n - 1$, then

$$\varphi_k[\mathfrak{A}^{(j)}, S_k(\mathfrak{A}^{(j)}, \mathfrak{A})] = \widetilde{\varphi}_k[\mathfrak{A}^{(j)}, S_k(\mathfrak{A}^{(j)}, \mathfrak{A})].$$

Theorems 4 and 5 indicate conditions under which φ_k and $\widetilde{\varphi}_k$ coincide on the given neighborhood $S_k(\mathfrak{A}^{(j)}, \mathfrak{A})$. Under these conditions, the computation of φ_k may be carried out on the basis of definition (11).

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$$^{*}\ \rho(\alpha, \beta) = \sum_{i=1}^{n} |\alpha_i - \beta_i| \quad (\alpha = \alpha_1 \dots \alpha_n),\ \beta = (\beta_1 \dots \beta_n).$$

Note: Figure translations are in progress. See original paper for figures.