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V. N. MONAKHOV

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Abstract

Full Text

MATHEMATICS

V. N. MONAKHOV

AN INVERSE MIXED BOUNDARY-VALUE PROBLEM FOR SEVERAL UNKNOWN ARCS

(Presented by Academician I. N. Vekua on 13 VII 1961)

Let on each known arc L_z^i $(i = 1, \dots, m)$ of the contour L_z the boundary condition be prescribed

$$\Phi_i(u, v) = 0, \tag{1}$$

relating the real and imaginary parts of the sought analytic function $w(z) = u + iv$, while on the unknown arcs $\overline{L}_z^i$ $(i = 1, \dots, m)$ $w(\tau)$ is prescribed,

$$u = f_1^i(\tau), \qquad v = f_2^i(\tau), \qquad \tau \in [\tau_n^i, \tau_1^{i+1}]. \tag{2}$$

It is required to determine the closed contour L_z, consisting of the given arcs L_z^i and the sought arcs $\overline{L}_z^i$, and the function $w(z)$, analytic inside the domain $D(L_z)$, satisfying conditions (1), (2). This problem, called the **inverse mixed boundary-value problem**, was solved in the author' s papers $(^{1,2})$ for the case of one unknown arc for various values of the parameter τ: $x = \operatorname{Re} z$, $\alpha = \operatorname{arc\,tg} \dfrac{dy}{dx}$, $\theta = \arg z$, s—the arc abscissa of the unknown arc. Various special cases of this problem were also investigated $(^3)$. The problem for the case of an infinite contour L_z is posed and solved in an entirely analogous way. Particular cases of the formulated problem are certain hydrodynamical problems: inverse problems of filtration theory $(^4)$, impact theory $(^5)$, and, in a certain sense, problems of flow past arcs with separation of jets (the case of an infinite contour L_z).

Let us suppose first that $\tau \equiv x = \operatorname{Re} z$, and that L_z^i $(i = 1, \dots, m)$ are polygons with number of sides equal to $n_i - 1$, and with a prescribed direction of traversal. Intersections of different intervals $[x_n^i, x_1^{i+1}]$ and the possibility of the inequality $x_n^i > x_1^{i+1}$ are allowed. The position of one of the polygons, for example L_z^1, is fixed relative to the origin of coordinates of the z-plane; the remaining polygons are prescribed up to a displacement along the OY axis. Obviously, depending on the traversal, each polygon L_z^i lies in the strip $x_1^i \le x \le x_n^i$ or $x_n^i \le x \le x_1^i$ (we omit the index i at n_i everywhere). We assume, of course, that the problem as a whole is posed geometrically correctly, i.e., that it is possible to connect the ends of the given polygons L_z^i by certain curves situated in the corresponding strips between neighboring polygons in such a way that the resulting contour preserves the direction of traversal on the given polygons and has no self-intersection points.

The contour L_w, consisting of arcs L_w^i with equations (1) and arcs $\overline{L}_w^i$ with equations (2), is assumed to be closed, piecewise smooth, and without self-intersection points. The functions $f_1^i(x)$, $f_2^i(x)$, and $\Phi_i(u,v)$ have derivatives with respect to their arguments satisfying the Hölder condition and are coordinated so that

the directions of traversal on the contours L_z and L_w coincide. Then, mapping the upper half-plane of the ζ-plane onto the domain $D(L_w)$, internal to the boundary L_w, and comparing the values of the obtained function $w = \omega(\zeta)$ with conditions (2), we obtain the boundary-value problem (cf. (1))

$$\begin{aligned} k_j^i \frac{dx}{dt} - \frac{dy}{dt} &= 0, \qquad t \in [t_j^i, t_{j+1}^i], \\ \frac{dx}{dt} &= h_i(t), \qquad t \in [t_n^i, t_1^{i+1}], \end{aligned} \tag{3}$$

where k_j^i $(j = 1, \dots, n-1;\ i = 1, \dots, m)$ are the tangents of the angles of inclination of the sides of the polygon L_z^i to the OX axis; t_j^i $(j = 1, \dots, n_i;\ i = 1, \dots, m)$ are the preimages of the vertices and endpoints of the polygons lying on the real axis of the ζ-plane.

The known functions $h_i(t)$ are representable in the form

$$h_i(t) = h_i^*(t)(t - t_n^i)^{\gamma_n^i - 1}(t - t_1^{i+1})^{\gamma_1^{i+1} - 1},$$

where $0 < \gamma_j^i < 2$ $(j = 1, n)$ and $0 \neq |h^*(t)| < \infty$ for $t \in [t_n^i, t_1^{i+1}]$.

Let us fix, for example, the points t_1^1, t_n^1, t_1^2 and assume that, under this, the infinitely distant point $t = \pm\infty$ passes into some point of the arc L_z^1, i.e. one of the intervals $[t_j^1, t_{j+1}^1]$ is infinite. From these conditions the function $w = \omega(\zeta)$, and consequently also the points t_n^i and t_1^i $(i = 1, \dots, m)$, will be determined uniquely. The canonical function of the homogeneous Hilbert problem with discontinuous coefficients, satisfying the necessary conditions at the angles of the polygons L_z^i, can be written in the form

$$\Pi(\zeta) = \Pi_*(\zeta) \prod_{k=1}^{m} (\zeta - t_n^k)^{\varepsilon_n^k} (\zeta - t_1^k)^{\varepsilon_1^k},$$

where

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$$\Pi_*(\zeta) = C_0 \prod_{k=1}^{m} \prod_{i=1}^{n_i} (\zeta - t_k^i)^{\alpha_k^i - 1}$$

is the derivative of an analytic function conformally mapping the domain $\operatorname{Im} \zeta \geq 0$ onto some finite polygon P_z, composed of polygons P_z^i with sides parallel to the sides L_z^i, and of segments $\bar{P}_z^i$ of straight lines parallel to the OY axis, joining the corresponding ends of neighboring polygons while preserving the traversal on them. The numbers $\varepsilon_1^k, \varepsilon_n^k$ are integers, and

$$\sigma = \sum_{k=1}^{m} (\varepsilon_n^k + \varepsilon_1^k) = 1, \qquad |\alpha_l^k - 1 + \varepsilon_l^k| < 1.$$

Representations of the solution of the nonhomogeneous Hilbert problem corresponding to the case $\sigma > 1$ reduce to one of the representations for which $\sigma = 1$.

The number χ of distinct representations of the solution depends on the number of polygons and their mutual arrangement, and therefore in the general case its computation is difficult. However, from the very construction of the canonical function it is clear that $\chi \geq 1$. By choosing one or another representation of the canonical function, one can dispose of the angles of junction of the polygons L_z^1 and the unknown arcs $\bar{L}_z$. The subsequent reasoning is carried out for some fixed canonical function $\Pi(\zeta)$.

By construction, $\Pi(\zeta)$ has at infinity a zero of first order; consequently, the general solution of problem (3), bounded at infinity,

is written in the form:

$$z = F(\zeta) = \sum_{i=1}^{m} \frac{\Pi(\zeta)}{\pi i} \int_{t_n^s}^{t_1^{i+1}} \frac{h_i(t)\,dt}{\Pi(t)(t-\zeta)}\, d\zeta + C, \tag{4}$$

where the limits in the inner integrals are fixed. Let us write the system for determining the constants that have remained arbitrary:

$$z_n^1 = F(t_n^1),$$

$$l_k^i = \int_{t_k^i}^{t_{k+1}{}^i} \left| \frac{dF}{dt} \right| dt \qquad (k = 2, n_i - 1;\ i = 1, \dots, n), \tag{5}$$

where l_k^i are the lengths of the sides of the corresponding polygons.

It is easy to show that this system completely determines the problem, i.e., if it is solvable with respect to t_k^i and C, then by formula (4), for $\zeta = t$, one obtains a contour containing the prescribed polygons L_z^i. The main difficulty in proving the existence of the constants lies in proving the local uniqueness of the solution of system (5).

Suppose that system (5) has at least two infinitely close solutions. From formula (4) we compute

$$\delta z = \sum_{k,i} \frac{\partial z}{\partial t_k^i}\, \delta t_k^i + \delta C.$$

On the other hand, one can show that $d\delta z/d\zeta$ satisfies, up to quantities of second order of smallness with respect to δt_k^i and δC, the boundary-value problem

$$k_j^i \frac{d\delta x}{dt} - \frac{d\delta y}{dt} = 0, \qquad t \in [t_j^i, t_{j+1}^i] \quad (i = 1, \dots, m),$$

$$\frac{d\delta x}{dt} = 0, \qquad t \in [t_n^i, t_1^{i+1}] \quad (i = 1, \dots, m). \tag{6}$$

Having found $d\delta z/d\zeta$ from the boundary-value problem (6) and comparing it with $d\delta z/d\zeta$ computed from formula (4), we become convinced that their equality is possible only when

$$d\delta z/d\zeta = 0$$

up to quantities of second order of smallness; whence, without difficulty, it follows that $\delta z = 0$. Further, by the method of continuity of Weinstein [6], the existence of the solution is proved from the local uniqueness of the solution; and from the uniqueness of the solution of the problem for the case when all polygons L_k^i degenerate into segments there follows uniqueness in the general case.

Thus, the existence and uniqueness of the function $z = F(\zeta)$ is established; from it the sought function $\omega(z) = \omega[F^{-1}(z)]$ is determined, where $\zeta = F^{-1}(z)$ is the function inverse to $z = F(\zeta)$.

By a limiting passage from polygons, the existence of a solution of the inverse mixed boundary-value problem is proved for the case of curvilinear arcs L_z^i, analogously to the case of one arc ([1], Theorem 2).

The case of other values of the parameter τ, as well as special cases, is considered analogously to the case of one arc (see [2,3]). We have also considered the case when the prescribed intervals $[x_n^i, x_1^{i+1}]$ are divided into a finite sum of intersecting intervals.

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