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Abstract

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MATHEMATICS

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NONTRIVIAL SOLUTIONS OF THE DIRICHLET PROBLEM FOR EQUATIONS WITH THE MONGE–AMPÈRE OPERATOR

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1. In the present work we consider the question of nonnegative solutions of the equation

$$rt - s^2 = f(x, y, z, p, q)(1 + p^2 + q^2)^{\alpha}, \tag{1}$$

where $0 \leqslant \alpha \leqslant 1$, satisfying the condition

$$z(x, y)\big|_{\Gamma} = 0. \tag{2}$$

The problem is considered in a bounded convex domain Ω, whose boundary Γ has curvature bounded below by a positive number. With respect to $f(x, y, z, p, q)$ it is assumed that it is continuous in the aggregate of variables $\{x, y\} \in \overline{\Omega}$, $z \geqslant 0$, $-\infty < p, q < \infty$, nonnegative, and, for values of z from any finite interval, uniformly bounded above with respect to the remaining variables. The nonnegativity of $f(x, y, z, p, q)$ means that equation (1) is elliptic and that the solutions of problem (1)–(2) (if they exist) are convex functions.

Each convex function $z(x, y)$ generates, by means of its supporting planes $z - z(x_0, y_0) = p(x - x_0) + q(y - y_0)$, a mapping of points $\{x_0, y_0\} \in \overline{\Omega}$ into the plane of the variables $\{p, q\}$. This mapping is called the normal mapping ([1]). The normal mapping is multivalued at edge and conical points of the function $z(x, y)$; the normal mapping generates mappings of sets.

By a solution of problem (1)–(2) we shall understand a nonnegative convex function with absolutely continuous area of the normal image, satisfying equation (1) almost everywhere and vanishing on Γ.

2. Let us first consider the problem

$$rt - s^2 = \varphi(x, y)(1 + p^2 + q^2)^{\alpha}, \qquad z(x, y)\big|_{\Gamma} = 0. \tag{3}$$

It turns out (see [1]; for the case $\alpha = 0$ this result was obtained by A. V. Pogorelov [2]) that problem (3) in the indicated class has a unique solution, if $\varphi(x, y)$ is summable. Let $z(x, y) = A_\alpha \varphi(x, y)$.

Theorem 1. *The operator A_α maps every uniformly bounded family of non-negative functions into a set compact in the sense of uniform convergence. The operator A_α maps every uniformly bounded and pointwise convergent sequence of functions into a uniformly convergent sequence.*

This theorem is proved by the method developed in [3].

Theorem 2. A. *The operators A_α are monotone in the sense that from $0 \leqslant \varphi(x, y) \leqslant \psi(x, y)$ there follows the inequality $A_\alpha \varphi(x, y) \leqslant A_\alpha \psi(x, y)$.*

B. For $0 \leq \alpha < 1$, for every nonnegative function $\varphi(x, y)$ the inequalities hold

$$A_\alpha[\lambda \varphi(x, y)] \geq \lambda^{\frac{1}{2(1-\alpha)}} A_\alpha \varphi(x, y) \qquad (0 \leq \lambda \leq 1). \tag{4}$$

C. If $\alpha_1 < \alpha_2$, then for every nonnegative $\varphi(x, y)$ the inequality $A_{\alpha_1} \varphi(x, y) \leq A_{\alpha_2} \varphi(x, y)$ holds.

D. The following estimates are valid

$$0 \leq A_\alpha \varphi(x, y) \leq \begin{cases} r_0 \sqrt{[(1-\alpha)\|\varphi\| r_0^2 + 1]^{\frac{1}{1-\alpha}} - 1}, & \text{if } 0 \leq \alpha < 1, \\ r_0 \sqrt{e^{\|\varphi\| r_0^2}}, & \text{if } \alpha = 1; \end{cases}$$

$$p^2 + q^2 \leq \begin{cases} [(1-\alpha)\|\varphi\| r_0^2 + 1]^{\frac{\alpha}{1-\alpha}} - 1, & \text{if } 0 \leq \alpha < 1, \\ e^{\|\varphi\| r_0^2} - 1, & \text{if } \alpha = 1, \end{cases}$$

where p and q are the angular coefficients of any supporting plane to the graph of the function $A_\alpha \varphi(x, y)$; $1/r_0$ is a lower estimate of the specific curvature of Γ.

3. Consider the operator

$$B\varphi(x, y) = A_\alpha f\left[x, y, \varphi(x, y), \frac{\partial}{\partial x}\varphi(x, y), \frac{\partial}{\partial y}\varphi(x, y)\right]. \tag{5}$$

Theorem 3. The operator B leaves invariant the cone of nonnegative convex functions satisfying condition (2), and is completely continuous on this cone (in the sense of the uniform metric).

It is not hard to see that the fixed points of the operator B coincide with the solutions of the boundary-value problem (1)–(2). This makes it possible, in proving existence theorems, to apply various general fixed-point principles in a cone [4,5].

4. **Theorem 4.** Suppose that the condition

$$f(x,y,z,p,q) \leq \begin{cases} a_1(1+z^2)^\gamma, & \text{if } 0 \leq \alpha \leq 1, \\ a_1 \ln^{1-\varepsilon}(2+z), & \text{if } \alpha = 1, \end{cases} \tag{6}$$

is satisfied, where $\gamma < 1-\alpha$, $\varepsilon > 0$, $a_1 > 0$. Then the boundary-value problem (1)–(2) has at least one solution.

If $f(x,y,0,0,0) \equiv 0$, then the boundary-value problem has the trivial zero solution. The question arises of conditions for the existence of nontrivial solutions.

Theorem 5. Suppose that condition (6) is satisfied and suppose there exists $\delta_0 > 0$ such that

$$f(x,y,z,p,q) \geq a_2 z^{2-\varepsilon} \qquad (0 \leq z \leq \delta_0;\ -\infty < p,q < \infty), \tag{7}$$

where $a_2 > 0$, $\varepsilon > 0$. Then the boundary-value problem (1)–(2) has at least one solution distinct from the identically zero one.

For the proof of this theorem one applies the principle of existence of a nonzero fixed point for operators that are contractions of a cone ([5]).

Uniqueness of the nontrivial solution has so far been proved for the case when $0 \leq \alpha < 1$ and

$$f(x,y,z,p,q) \equiv f(x,y,z). \tag{8}$$

Theorem 6. Suppose $f(x,y,z)$ is nondecreasing in z; $f(x,y,z) > 0$ for $z > 0$ and almost all points $\{x,y\} \in \Omega$. Suppose

$$f(x,y,\lambda z) \geq \lambda^{\gamma_0} f(x,y,z) \qquad (\{x,y\} \in \Omega;\ 0 \leq \lambda \leq 1;\ z \geq 0), \tag{9}$$

where $\gamma_0 < 2(1-\alpha)$. Then the boundary-value problem (1)–(2) can have more than one nonnegative solution distinct from the identically zero solution.

In the proof of this theorem the u_0-concavity [(4)] of the operator B is used, where $u_0(x,y)$ is the function whose graph is a cone with vertex over some interior point of the domain Ω, whose directrix is the contour Γ.

It is interesting to note that the uniqueness of the nontrivial solution in the conditions of Theorem 6 is obtained under assumptions for which there is no maximum principle.

5. In Theorems 4–6 it was assumed that $f(x,y,z,p,q)$, as a function of the variable z, grows essentially more slowly than z^2. We give a criterion for the existence of a nontrivial solution that covers the case of arbitrarily "strong" nonlinearities in z.

Theorem 7. *Suppose there exist $\delta_0 > 0$ and $M_0 > 0$ such that*

$$f(x,y,z,p,q) \leq a_3 z^{\gamma_1} \quad (\{x,y\} \in \Omega;\ 0 \leq z \leq \delta_0;\ -\infty < p,q < \infty); \tag{10}$$

$$f(x,y,z,p,q) \geq a_4 z^{\gamma_1} \quad (\{x,y\} \in \Omega;\ z \geq M_0;\ -\infty < p,q < \infty), \tag{11}$$

where $\gamma_1 > 2$, $a_3, a_4 > 0$. Then the boundary-value problem (1)–(2) has, besides the trivial zero solution, at least one more solution.

Under the conditions of this theorem the operator B is an expansion of the cone [(5)].

6. In the case when the function $f(x,y,z,p,q)$, as a function of the variable z, has alternating intervals of rapid growth and intervals of slow growth, the boundary-value problem (1)–(2) has, generally speaking, many solutions. We indicate one particular theorem for the case $0 < \alpha < 1$.

Theorem 8. *Suppose there exists a sequence $R_n \to \infty$ such that*

$$f(x,y,z,p,q) \geq a z^{\gamma_1} \quad (\delta R_n \leq z \leq R_n),$$

where $\gamma_1 > 2$ and $\delta > 0$ is a sufficiently small number. Suppose, further, that there exists a sequence $R_n^* \to \infty$ such that

$$f(x,y,z,p,q) \leq a_n (1+z^2)^{\gamma_2} \quad (0 \leq z \leq R_n^*),$$

where $\gamma_2 < 1-\alpha$, and the numbers a_n satisfy the inequalities

$$r_0 \sqrt{\left[(1-\alpha) a_n (1+R_n^{*2})^{\gamma_2} r_0^2 + 1\right]^{\frac{1}{1-\alpha}} - 1} < R_n^*,$$

where $1/r_0$ is a lower estimate for the specific curvature of the contour Γ. Then the boundary-value problem (1)–(2) has a countable number of distinct solutions z_n, whose maxima increase without bound as $n \to \infty$.

7. In [(6)] theorems are established from which it follows that the operator

$$A_z[E(x,y,z,p,q)r + 2F(x,y,z,p,q)s + G(x,y,z,p,q)t]$$

is completely continuous on the cone K of convex functions and leaves this cone invariant if

$$0 \leq -E(x,y,z,p,q)\xi^2 - 2F(x,y,z,p,q)\xi\eta - G(x,y,z,p,q)\eta^2 \leq \frac{\xi^2+\eta^2}{Q_1(p)+Q_2(q)},$$

where $Q_1(p) > 0$, $Q_2(q) > 0$, and

$$\int_{-\infty}^{\infty} \frac{dp}{Q_1(p)} < \infty, \qquad \int_{-\infty}^{\infty} \frac{dp}{Q_2(p)} < \infty.$$

This makes it possible to extend the theorems formulated above to the case of equations

$$\frac{rt - s^2}{(1+p^2+q^2)^\alpha} = E(x,y,z,p,q)r + 2F(x,y,z,p,q)s +$$

$$+ G(x,y,z,p,q)t + f(x,y,z,p,q)$$

with a quasilinear part.

The formulated theorems also admit a generalization to the case of equations

$$\frac{rt - s^2}{R(p,q)} = f(x,y,z,p,q),$$

where $R(p,q)$ is distinct from $(1+p^2+q^2)^\alpha$ $(0 \leqslant \alpha \leqslant 1)$.

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Note: Figure translations are in progress. See original paper for figures.