Mathematics

1961

SovietRxiv

Abstract

Full Text

Mathematics

V. F. Krotov

The Fundamental Problem of the Calculus of Variations for the Simplest Functional on a Class of Discontinuous Functions

(Presented by Academician A. N. Kolmogorov on 5 IX 1960)

We consider the problem of the extremum of the functional

$$I(u) = \int_a^b F(x, y, y')\,dx \tag{1}$$

on a class U of curves possessing the following properties: 1) continuity of the curve, i.e., the coordinates x and y of the points of the curve u can be represented as continuous functions of some parameter t; 2) the function $y(x)$ is continuous and single-valued everywhere on $[a, b]$, except for a finite set of points x_i $(i = 1, 2, \dots, k)$, where it may have discontinuities of the first kind; 3) the derivative $y'(x)$ of the function $y(x)$ is continuous and bounded on the intervals $(a, x_i), \dots, (x_i, x_{i+1}), \dots, (x_k, b)$; 4) the function $y(x)$ satisfies the condition $y(a) = a1,\ y(b) = b1$; 5) there exists a simply connected closed domain B of the XY-plane in which $F(x, y, z)$ is continuous in all three arguments together with the derivatives F_x, F_y, F_z for all z, and all curves of the set U lie in this domain; 6) everywhere in the domain B the existence is assumed of the limits

$$W(x, y, \operatorname{sign} m) = \lim_{m \to \pm\infty} \frac{1}{m} F(x, y, m), \tag{2}$$

uniformly bounded in every finite subdomain of the domain B.

The paper gives a definition of the functional (1) on the class U and proves a number of theorems very useful for its study.

Definition 1. By the **functional** (1) **of a curve** $u \in U$ we shall mean the limit

$$I(u) = \lim_{m \to \infty} I(u^m), \tag{3}$$

where u^m is the curve u in which all vertical segments are inclined at an angle $1/m$ to the vertical so that the function $y(x)$ along u^m is single-valued. From

Fig. 1

Figure 1: Fig. 1

the latter condition, taking $m > 0$ if the inclination is made clockwise, we obtain $\operatorname{sign} m = \operatorname{sign}(y_i - \bar{y}_i)$, where $y_i = y(x_i + 0)$, $\bar{y}_i = y(x_i - 0)$. Carrying out in (3) the limiting transition, we shall have

$$I(u) = \sum_{i=1}^{k} \int_{\bar{y}_i}^{y_i} W[x_i, \xi, \operatorname{sign}(y_i - \bar{y}_i)]\, d\xi + \sum_{i=1}^{k-1} \int_{x_i+0}^{x_{i+1}-0} F(x, y, y')\, dx$$
$$+ \int_{a+0}^{x_1-0} F\, dx + \int_{x_k+0}^{b-0} F\, dx + \int_{a_1}^{y(a+0)} W(a, \xi)\, d\xi + \int_{y(b-0)}^{b_1} W(b, \xi)\, d\xi. \tag{4}$$

From the properties of the function $F(x, y, z)$ and of the class U it follows that the functional exists on every curve $u \in U$.

Let us introduce into consideration the set of broken lines $\{r_n\} \subset U$, represented by the functions $y_n(x) = y_i + y_i'(x - x_i)$ for $x_i < x < x_{i+1}$ $(i = 0, 1, \ldots, n-1)$,

where $a = x_0 < x_1 < \cdots < x_i < \cdots < x_{n-1} < x_n = b_1$ are the abscissas of the points of discontinuity of the function $y_n(x)$, and y_i and y_i' are $2n$ independent parameters defining the broken line γ_n (see Fig. 1).

Definition 2. Let $u \in U$ be given by the function $y(x)$. We shall say that $\gamma_n \to u$, and correspondingly $y_n(x) \to y(x)$, if

$$y_i = y(x_i + 0); \qquad y_i' = y'(x_i + 0);$$

$$|x_{i+1} - x_i| < \varepsilon$$

for $n > N$. Here $i = 0, 1, \ldots, n$; $\varepsilon > 0$. In this sense $\{\gamma_n\}$ is everywhere dense in U.

Fig. 1

Definition 3. Let two functions be given: $y^0(x) \in U_n$ and $z(x)$, bounded and continuous on the intervals $(a, x_1), \ldots, (x_i, x_{i+1}), \ldots, \ldots, (x_k, b)$, and let $\{\overline{\gamma}_n\} \subset \{\gamma_n\}$ be a sequence of broken lines such that

$$y_i = y^0(x_i + 0); \quad y_i' = z(x_i + 0); \quad |x_{i+1} - x_i| < \varepsilon \tag{5}$$

for $n > N$. Here $i = 0, 1, \ldots, n$; $\varepsilon > 0$.

In this case we shall say that $\overline{\gamma}_n \to u_0 \in U_0$. We shall call the limit u_0 a (y^0, z)-line. The totality U_0 will be called the closure of the set $\{\gamma_n\}$. A (y^0, z)-line $u_0 \in U$, if $y^{0\prime}(x) = z(x)$ everywhere on $[a, b]$, except at the points x_i. Consequently, $U \subset U_0$.

Definition 4. Define the functional (1) of the line $u_0 \in U_0$ as

$$I(u_0) = \lim_{\gamma_n \to u_0} I\{\gamma_n\}. \tag{6}$$

Theorem 1. *If the right and left limits (2) exist everywhere in the domain $B(x, y)$, and moreover $W(x, y, 1) = W(x, y, -1)$, then the functional (6) exists everywhere on U_0 and can be represented in the form*

$$I(u_0) = (R) \int_a^b S(x, y^0, z)\, dx + \Phi(b, b_1) - \Phi(a, a_1), \tag{7}$$

where

$$S(x, y^0, z) = F(x, y^0, z) - W(x, y^0)z - \int_{c(x)}^{y^0} W_x(x, \xi)\, d\xi + W[x, c(x)]\, c'(x), \tag{8}$$

$$\Phi(x, y) = \int_{c(x)}^{y} W(x, \xi)\, d\xi; \tag{9}$$

$c(x)$ is an arbitrary smooth function.

Indeed, using (4) and grouping the terms in (4) in a different way, we may write:

$$I(\gamma_n) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) S(x_i^*, y(x_i^*), z(x_i^*)) + \Phi(b, b_1) - \Phi(a, a_1), \tag{10}$$

where

$$S = \frac{1}{x_{i+1} - x_i} \left\{ \int_{x_i}^{x_{i+1}} F(x, y_i + y_i'(x - x_i), y_i')\, dx + \Phi(x_i, y_i) - \Phi(x_{i+1}, y_i + y_i'(x_{i+1} - x_i)) \right\}.$$

or, by Lagrange' s theorem on finite increments:

$$S(x_i^*, y^0(x_i^*), z(x_i^*)) = F[x_i^*, y^0(x_i^*), z(x_i^*)] - W[x_i^*, y^0(x_i^*)] z(x_i^*) +$$

$$+W[x_i^*, c(x_i^*)]c'(x_i^*) - \int_c^{y^0(x_i^*)} W_x(x_i^*, \xi)\, d\xi + o(\varepsilon), \qquad x_i < x_i^* < x_{i+1}.$$

By the properties of the function $F(x, y, z)$, the function $S(x, y, z)$ is finite and continuous in all three arguments in the domain $B(x, y)$ for arbitrary values of z. Since the functions $y^0(x)$ and $z(x)$ are finite and almost continuous on $[a, b]$, $S(x, y^0(x), z(x))$ is also almost continuous, and, consequently, by Lebesgue' s theorem, the function $S(x, y^0(x), z(x))$ is Riemann integrable, i.e., as $\gamma_n \to u_0$, the sum (10) has the limit (7), independent of the choice of the sequence γ_n. The theorem is proved. Theorem 2 is proved similarly.

Theorem 2. *If the right and left limits (2) exist everywhere in the domain $B(x, y)$, but $W(x, y, 1) \neq W(x, y, -1)$, then the functional (1) exists everywhere on U_0 and can be represented in the form*

$$I(u_0) = \sum_{i=1}^{k-1} (R) \int_{x_i+0}^{x_{i+1}-0} [F(x, y, z) + W(x, y, \operatorname{sign}(y' - z))(y' - z)]\, dx +$$

$$+ \sum_{i=1}^{k} \int_{\bar{y}_i}^{y_i} W(x_i, \xi, \operatorname{sign}(y_i - \bar{y}_i))\, d\xi + \int_{a+0}^{x_1-0} [F + W(x' - z)]\, dx +$$

$$+ \int_{x_k+0}^{b-0} [F + W(y' - z)]\, dx. \tag{11}$$

If an additional condition is imposed on U_0, namely that the equality

$$\operatorname{sign}(y^{0'} - z) = \operatorname{sign}(\bar{y}^{0'} - \bar{z}),$$

where $y^{0'}(x) = y^0(x + 0)$, $\bar{y}^{0'}(x) = y^{0'}(x - 0)$, $z = z(x + 0)$, $\bar{z} = z(x - 0)$, is violated only at a finite number of points $x = \mu_j \in [a, b]$ $(j = 1, 2, \dots, r)$, then we shall also have

$$i(u_0) = (R) \int_a^b S[(x, y^0, z, \operatorname{sign}(y^{0'} - z))]\, dx + \sum_{j=1}^{r} [\Phi(\mu_j, y_j, \operatorname{sign}(y_j' - z_j)) -$$

$$-\Phi(\mu_j, y_j, \operatorname{sign}(\bar{y}_j - \bar{z}_j))] + \Phi[b, b_1, \operatorname{sign}(b_1 - y(b))] -$$

$$-\Phi[a, a_1, \operatorname{sign}(y(a) - a_1)]. \tag{12}$$

From formulas (7), (11), (12) there immediately follow certain properties of the functional (1) on the set U_0: 1) the functional $I(u_0)$, $u_0 \in U_0$, depends on two independent functions $y^0(x)$ and $z(x)$, whose derivatives either do not enter the integrand at all, as in (7), or enter only in the form $\operatorname{sign}(y' - z)$, as in (12); 2) directly from (7) and (11) follows the continuity of $I(u_0)$ on the set U_0 in the sense that $|I(u_0^*) - I(u_0)| < \varepsilon$, if $|y_0^*(x) - y^0(x)| < \eta$ and $|z^*(x) - z(x)| < \eta$ everywhere on $[a, b]$, except for an η-neighborhood of the discontinuity points $x = x_i$; 3) since $U \subset U_0$ and the functional (1) is continuous on the set U_0, in the class U definitions 1 and 4 of the functional (1) coincide.

It should also be noted that expressions (7) and (12) are valid for functionals of a broader class as well, namely for functionals of the form (1) in which the function $F(x, y, z)$ may have discontinuities of its derivatives.

The following is also easily proved.

Theorem 3. *Let* $\inf I(u_0) = m > -\infty$, $u_0 \in U_0$, *and let the line* v *be a minimum of the functional (1) in the class* U_0, *i.e.* $I(v) = m$. *Then* v

there is also a minimum in the class U, i.e. $\inf\limits_{u \in U} I(u) = I(v)$. Conversely, if v is a minimum of the functional (1) in the class U, then it is also a minimum in the class U_0.

Directly from expression (14) and Theorem 3 there follows

Theorem 4. If the conditions under which Theorems 1 and 3 are valid hold, then in order that the functional (1) have on the line $u_0 \in U_0$ an absolute or relative extremum in the class of lines U and U_0, it is necessary and sufficient that, for each fixed $x \in [a, b]$, the function of two variables $S(x, y^0, z)$ have the corresponding extremum. More precisely:

$$S(x, y^0, z) = \inf S(x, y^0, z), \qquad y^0 \in B, \quad -\infty < z < \infty, \tag{13}$$

if, for example, an absolute minimum is in question.

Corollary 1. From condition (13) it follows as a necessary condition that the extremal (y^0, z)-line $u_0 \in U_0$ may consist of continuous pieces satisfying the equations

$$S_y \equiv F_y(x, y^0, z) - W_y(x, y^0)z - W(x, y^0) = 0; \qquad S_z \equiv F_z - W = 0, \tag{14}$$

and of pieces of the boundary Γ of the domain B, joined by vertical segments with the points (a, a_1) and (b, b_1) and with one another at $x = x_i$, where the condition

$$S|_{x_i - 0} = S|_{x_i + 0}. \tag{15}$$

is satisfied.

The functions $y^0(\bar{x})$ and $\bar{z}(x)$ satisfying (13) define an extremal line $u_0 \in U_0$. If it turns out a posteriori that $\bar{y}^{0\prime}(x) = \bar{z}(x)$ almost everywhere on $[a, b]$, then the extremal u_0 of the functional (1) on the aggregate U itself belongs to U. In this case it follows from (14) that almost everywhere on $[a, b]$ the extremal satisfies the Euler equation.

Thus, if $W(x, y, \pm 1)$ exist and are equal everywhere in the domain $B(x, y)$, then the functional (1) attains a maximum or a minimum in the class of lines U on extremals of a special type (type a)), fundamentally different from Euler-Lagrange extremals. In contrast to the latter, these extremals are not solutions of a boundary-value problem for a differential equation; each infinitely small element of them is independent of its neighbors and possesses the property of a maximum or a minimum. The finite equation of such an extremal is written directly in the form of the necessary condition (14). The extremal of the functional (1) on the aggregate U itself belongs either to this aggregate or to the broader U_0. The latter means that if one takes the broken line γ_n described above, setting $y_i = \bar{y}^0(x_i)$, $y_i' = \bar{z}(x_i)$ $(i = 1, 2, \dots, n)$, where $\bar{y}^0(x)$, $\bar{z}(x)$ is a solution of (13), then, for sufficiently large n, $I(\gamma_n) < I(u)$, where $u \in U$ is any line, if, for example, a minimum is in question.

Using Theorems 2 and 3, one can also show (for lack of space we shall not dwell on this) that if the functions $W(x, y, \pm 1)$ exist and are not equal everywhere in the domain B, then a relative extremum of the functional (1) in the class U may be attained both on the extremals of type a) described above and, under certain additional conditions, on ordinary Euler extremals in the class of functions continuous on (a, b) (extremals of type b)).

The results set forth have a number of interesting applications in mechanics.

Remark. Extremals of type a in the class U may or may not be extremals in the class $\bar{U}$ of lines given in parametric form, depending on whether the lines $x = \text{const}$ are extremals in $\bar{U}$ or merely a boundary direction for the aggregate of lines along which the function $y(x)$ is single-valued.

Moscow Higher Technical School
named after N. E. Bauman

Received
21 X 1960

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.