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Abstract

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PLANE NONSTATIONARY MOTIONS OF AN IDEAL INCOMPRESSIBLE FLUID*

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The questions of existence and uniqueness of the solution of the Cauchy problem and of a mixed problem for the equations of motion of an ideal incompressible fluid have been investigated in ([1,2]). However, in these works the existence of a solution was proved only for a sufficiently small interval of time and in the absence of vortical mass forces. The difficulty of solving the problem "in the large" was pointed out, for example, in ([3]). In the present work, for the case of two-dimensional flows, the existence of a unique solution of the above-mentioned problems is established for all values of time $t \geq 0$, and without any assumptions on the smallness of the functions and parameters appearing in the conditions.

Thus, it is required to determine the velocity vector $\mathbf{v}(x,t)$ and the pressure $p(x,t)$ $(x = (x_1, x_2) \in \Omega)$ from the conditions:

$$\mathbf{v}_t + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla p + \mathbf{F}(x,t); \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0; \tag{2}$$

$$v_n\big|_S = \mathbf{v} \cdot \mathbf{n}\big|_S = 0; \tag{3}$$

$$\mathbf{v}\big|_{t=0} = \mathbf{a}(x). \tag{4}$$

Here Ω is the domain occupied by the fluid; S is its boundary with outward normal $\mathbf{n}$; $\mathbf{F} = (F_1, F_2)$ is a prescribed vector of mass forces; $\mathbf{a}$ is the initial velocity, assumed given. If the domain Ω is unbounded, then the condition that the velocity and pressure vanish at infinity is added.

Consider the case of a bounded domain Ω, since the general case is exhausted by similar considerations. Suppose the following conditions are satisfied: 1) Ω is a bounded domain of the plane (x_1, x_2); its boundary S consists of closed, twice continuously differentiable contours $S_0, S_1, \dots, S_n$, where $S_1, S_2, \dots, S_n$ lie inside the domain bounded by S_0; 2) $\mathbf{F}$ is a continuous vector and admits continuous derivatives with respect to x, t, F_{ix_k} $(i, k = 1, 2)$; 3) $\mathbf{a}$ has bounded first generalized derivatives. We consider the case of a simply connected domain ($S = S_0$). The changes in the proof entailed by dropping this condition are indicated below.

Introduce the stream function $\psi(x,t)$ by the equalities

$$v_1 = \psi_{x_2}, \qquad v_2 = -\psi_{x_1}. \tag{5}$$

To determine ψ we obtain the problem

$$\Delta\psi_t + \psi_{x_2}\Delta\psi_{x_1} - \psi_{x_1}\Delta\psi_{x_2} = f(x,t) \quad (a); \quad \psi\big|_S = 0 \quad (b); \quad \psi\big|_{t=0} = \varphi(x) \quad (c). \tag{6}$$

Here $f = F_{1x_2} - F_{2x_1}$; φ is the stream function of the vector $\mathbf{a}$.

Introduce some function spaces: V is the closure of the set of twice boundedly differentiable functions on $\Omega \times [0,T]$, equal to 0 on S, in the norm**

$$\varphi\|_V = \max_{x\in\Omega} |\Delta\varphi|;$$

V_1 is the space of functions of x, t defined in the cylinder $Q_T = \Omega \times [0,T]$ ($T > 0$ is some

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** By max we shall everywhere mean the essential maximum.

number), for almost all t, belonging as a function of x to the space V with norm $\|\psi\|_{V_1} \max_{0\le t\le T} \|\psi\|_V$; L_k–with norm $\|\varphi\|^k_{L_k} = \int_\Omega |\varphi|^k\, dx$. The space C' is the closure of the set of smooth functions defined in Q_T and equal to 0 on S, in the norm $\|\psi\|_{C'} = \max_{x,t}(|\psi_{x_1}| + |\psi_{x_2}|)$.

Lemma 1. Every function $\varphi \in V$ possesses second generalized derivatives with respect to x_1, x_2, belonging to any L_k ($k > 1$), and for large k the estimate holds

$$\sum_{i,j=1}^{2} \|\varphi_{x_i x_j}\|_{L_k} \le Ck\|\varphi\|_V, \tag{7}$$

where C is a constant independent of φ, k. The first derivatives of the function φ satisfy in Ω a Hölder condition with any exponent $0 \le \lambda < 1$.

The proof of inequality (7) uses some results on singular integrals in L_k, obtained in ([4]). We note that from (7) follows the summability of the functions $e^{\alpha|\varphi_{x_i x_j}|}$ for some $\alpha > 0$.

Define a generalized solution of problem (6) in Q_T ($T > 0$ arbitrary) as a function $\psi(x,t) \in V_1$ satisfying the integral identity

$$\int_\Omega \Delta\psi\,\Phi\,dx\Big|_{t=T} - \int_\Omega \Delta\varphi\,\Phi\,dx + \int_0^T \int_\Omega [-\Delta\psi\,\Phi_t - \Delta\psi(\Phi_{x_1}\psi_{x_2} - \Phi_{x_2}\psi_{x_1})]\,dx\,dt =$$

$$= \int_0^T \int_\Omega f\Phi\,dx\,dt \quad \text{for any function } \Phi(x,t) \text{ smooth in } Q_T. \tag{8}$$

Lemma 2. Let $\theta(x)$ be a solution of the boundary-value problem $\Delta\theta = g_{1x_1} + g_{2x_2}$; $\theta|_S = 0$, where g_1, g_2 are smooth functions in Ω. Then the estimate

$$\|\varphi_{x_1}\|_{L_k} + \|\varphi_{x_2}\|_{L_k} \le C_1(k)(\|g_1\|_{L_k} + \|g_2\|_{L_k})$$

is valid, where C_1* depends only on the domain, and for large k, $C_1(k) < C_2 k$ (C_2 does not depend on g_1, g_2, k).

Lemma 3. If ψ is a generalized solution of problem (6), then it has generalized derivatives $\psi_{x_1 t}, \psi_{x_2 t}$, and for any k

$$\max_{0\le t\le T} \|\psi_{x_i t}\|_{L_k} < C_3.$$

For the proof we apply Lemmas 1 and 2 to problem (6), written in the form

$$\Delta\psi_t = f - (\Delta\psi\,\psi_{x_2})_{x_1} + (\Delta\psi\,\psi_{x_1})_{x_2}; \qquad \psi_t|_S = 0.$$

Lemma 4. Problem (6) cannot have two generalized solutions in the sense of (8).

Let ψ_1, ψ_2 be generalized solutions and $\alpha = \psi_1 - \psi_2$. From (8) we easily obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\alpha_{x_1}^2 + \alpha_{x_2}^2)\,dx - \int_\Omega [\psi_{1x_1x_2}(\alpha_{x_1}^2 - \alpha_{x_2}^2) + (\psi_{1x_2x_2} - \psi_{1x_1x_1})\alpha_{x_1}\alpha_{x_2}]\,dx = 0. \tag{9}$$

From Lemma 1 follows the estimate $\alpha_{x_1}^2 + \alpha_{x_2}^2 < C_4^2$ for $x, t \in Q_T$. From (9), putting

$$z^2(t) = \int_\Omega (\nabla\alpha)^2\,dx,$$

we obtain

$$z\frac{dz}{dt} \le \int_\Omega (|\psi_{1x_1x_2}| + |\psi_{1x_2x_2} - \psi_{1x_1x_1}|)(\alpha_{x_1}^2 + \alpha_{x_2}^2)\,dx \le$$

$$\le C_5 C_4^\varepsilon \left(\|\psi_{x_1x_2}\|_{L_{2/\varepsilon}} + \|\psi_{1x_2x_2} - \psi_{1x_1x_1}\|_{L_{2/\varepsilon}}\right) z^{2-\varepsilon} \le \frac{C_6}{\varepsilon} C_4 z^{2-\varepsilon}, \quad z(t) \le C_4(C_6 t)^{1/\varepsilon}. \tag{10}$$

* C_i everywhere below denotes a constant depending only on the domain.

Let $0 \leqslant t \leqslant 1/2C_6$. Then, putting $\varepsilon \to 0$ in (10), we find that $z(t) = 0$. Repeating the preceding arguments, we find that $z(t) = 0$ for $t \in [1/2C_6,\, 1/C_6]$, $[1/C_6,\, 3/2C_6]$, etc. Thus, $z(t) = 0$ for $0 \leqslant t \leqslant T$ and, consequently, $\psi_1 = \psi_2$. The lemma is proved.

To prove existence, we shall show that a generalized solution satisfies a certain operator equation. For functions $\psi \in C'$ define the operator $A\psi$ by the equality

$$\int_\Omega (A\psi)\Phi\,dx\bigg|_{t=T} - \int_\Omega \Delta\varphi\,\Phi\,dx + \int_0^T\int_\Omega \left[-\Delta(A\psi)\Phi_t - \Delta(A\psi)(\Phi_{x_1}\psi_{x_2} - \Phi_{x_2}\psi_{x_1})\right]\,dx\,dt = \int_0^T\int_\Omega f\Phi\,dx\,dt, \tag{11}$$

where Φ is any smooth function defined on Q_T. Comparing (11) with (8), we conclude that every generalized solution of problem (6)

$$\psi = A\psi. \tag{12}$$

Lemma 5. *The operator A maps C' into the sphere of the space V_1*

$$\|A\psi\|_{V_1} \leqslant R = \|\varphi\|_\nu + \int_0^T \max_x |f(x,\tau)|\,d\tau. \tag{13}$$

and is completely continuous in C'.

From (13) it is easy to derive that every sphere of radius $\geqslant R_1$ in the space C' is mapped by the operator A into its compact part. By Schauder' s principle ([5]) we obtain that equation (12) has at least one solution ψ, with $\psi \in V_1$ and $\|\psi\|_\nu \leqslant R$. We note that, in order to obtain the estimate (13) in the case of sufficient smoothness of ψ, $\psi' = A\psi$, one must put in (11) $\Phi = (\Delta\psi')^{2k-1}$ and let $k \to \infty$. In the general case the proof is carried out by approximating the function $\psi \in C'$ by smooth functions.

Lemma 6. *The pressure $p(x,t)$, defined by equalities (1), (5), (6), is a bounded function; $\nabla p \in L_k$ for any $k > 1$.*

The pair $\mathbf{v}(x,t) = (\psi_{x_2}, -\psi_{x_1})$, $p(x,t)$ will be called a **generalized solution** of problem (1)–(4).

Let us pass to the case of a multiply connected domain. Define the functions $\psi_k(x)$ $(x \in \Omega)$ by the conditions $\Delta\psi_k = 0$; $\psi_k|_{S_r} = \delta_{kr}$; $k, r = 1, 2, \dots, n$, where δ_{kr} is the Kronecker symbol. Denote $\mathbf{u}_k = (\psi_{kx_2}, -\psi_{kx_1})$. Again introduce the stream function ψ by equality (5). ψ satisfies equation (6) and the initial condition (6). Instead of (6) we obtain the boundary condition $\psi|_{S_0} = 0$; $\psi|_{S_k} = \lambda_k(t)$ $(k = 1, 2, \dots, n)$, where $\lambda_k(t)$ are unknown functions. Put

$$\psi = \psi_0 + \sum_{k=1}^{n} \lambda_k(t)\psi_k; \qquad \varphi = \varphi_0 + \sum_{k=1}^{n} \lambda_{k0}\psi_k,$$

where $\lambda_{k0} = \varphi|_{S_k}$. Then ψ_0 is the solution of the problem

$$\Delta\psi_{0t} + \psi_{0x_2}\Delta\psi_{0x_1} - \psi_{0x_1}\Delta\psi_{0x_2} + \sum_{k=1}^{n} \lambda_k(\psi_{kx_2}\Delta\psi_{0x_1} - \psi_{kx_1}\Delta\psi_{0x_2}) = f;$$

$$\psi_0|_S = 0; \qquad \psi_0|_{t=0} = \varphi_0. \tag{14}$$

The requirement of uniqueness of $p(x,t)$ leads to n conditions

$$\sum_{k=1}^{n} \lambda_k'(t) \int_\Omega \mathbf{u}_k\mathbf{u}_r\, dx - \sum_{k=1}^{n} \lambda_k(t) \int_\Omega \mathbf{u}_k \times \operatorname{rot} \mathbf{v}_0\, \mathbf{u}_r + \int_\Omega \{\mathbf{v}_{0t} - \mathbf{v}_0 \times \operatorname{rot} \mathbf{v}_0 - \mathbf{F}\}\mathbf{u}_r\, dx = 0;$$

$$\mathbf{v}_0 = (\psi_{0x_2}, -\psi_{0x_1}); \qquad \lambda_r(0) = \lambda_{r0} \quad (r = 1, 2, \dots, n). \tag{15}$$

The generalized solution of problem (15) for fixed λ_k is determined analogously to the preceding one (in (11) one must replace ψ by ψ_0, and ψ_{x_i} by $\psi_{0x_i} + \sum_{k=1}^{n} \lambda_k\psi_{kx_i}$). Arguing as before, we determine ψ_0 as an operator of $(\lambda_1, \lambda_2, \dots, \lambda_n)$, after which we determine λ_k from (15). In doing this, besides an estimate of type (13), the energy equation is used in an essential way:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\nabla\psi)^2\, dx - \int_\Omega f\psi\, dx = 0. \tag{16}$$

The pressure is determined with the aid of (1). We obtain a generalized solution with the same differential properties as in the case of a simply connected domain.

Theorem 1. Suppose that conditions (1), (2), (3) are satisfied. Then problem (1)–(4) has, and moreover has a unique, generalized solution $\mathbf{v}(x,t)$, $p(x,t)$ $(x \in \Omega,\ 0 \le t < \infty)$. Moreover: 1) $\mathbf{v}, p$ are continuous in x, t and satisfy the Hölder condition with any $0 \le \lambda < 1$ in Ω; 2) $\max_{x,t\in Q_T} |\operatorname{rot} \mathbf{v}| < C_7(T)$; 3) the quantities $\|v_{ix_k}\|_{L_k}$, $\|\mathbf{v}_t\|_{L_k}$, $\|\nabla p\|_{L_k}$ are continuous in t; 4) equations (1), (2) are satisfied for all values of t almost everywhere in Ω; 5) conditions (3), (4) are fulfilled in the classical sense.

It is further possible to study the differential properties of the generalized solution obtained and to establish that it admits any number of derivatives with respect to x, t, continuous in the closed domain Q_T, if $\mathbf{F}, S, \mathbf{a}$ are sufficiently smooth. Namely, the following is true:

Theorem 2. Suppose: 1) the boundary S is continuously differentiable $r+1$ times; 2) $\mathbf{F}$ has r-th generalized derivatives[6] with respect to x, t, and their norms in L_k $(k > 2)$ are bounded in t; 3) $\mathbf{a} \in W_k^{(r)}$. Then $\mathbf{v}, p$ have r-th generalized derivatives with respect to x_i, t, and their norms in L_k are bounded in t.

In particular, for $r = 2,\ k > 2$ we obtain a classical solution of problem (1)–(4). The proof of this theorem uses the following fact:

Lemma 7. Let the boundary S be three times continuously differentiable, ψ be the solution of the boundary-value problem

$$\Delta\psi = g(x); \qquad \psi|_S = 0.$$

Then $g \in W_r^{(1)}$ $(r > 2)$ and the estimate $\max_{x\in\Omega} |g(x)| < C_8$ is known. Then the inequality

$$\max_{x\in\Omega} |\psi_{x_i x_k}| \le C_9 \ln\left[C_{10} + C_{11}(\|g_{x_1}\|_{L_r} + \|g_{x_2}\|_{L_r})\right]$$

is valid.

Remarks. 1. The requirements on the differential properties of $\mathbf{F}, \mathbf{a}$ in Theorems 1 and 2 can be somewhat weakened. 2. The method makes it possible to consider the case when $v_n|_S = \gamma$. 3. With the aid of the a priori estimates obtained, one can, by known schemes, justify the convergence of the Galerkin method and of finite-difference methods for the approximate computation of generalized solutions. 4. Equations analogous to (1)–(4), (6), occurring in certain questions of magnetohydrodynamics and the theory of diffusion in a fluid, can be investigated in the same way as was done above.

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Note: Figure translations are in progress. See original paper for figures.