MATHEMATICS

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1961

SovietRxiv

Abstract

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ON THE EMBEDDABILITY OF LOCALLY NORMAL GROUPS IN DIRECT PRODUCTS OF FINITE GROUPS

(Presented by Academician A. I. Mal' cev on 20 XII 1960)

Every subgroup of a direct product of finite groups is locally normal. However, it is known that subgroups of such a product do not exhaust all locally normal groups. A number of works have been devoted to conditions under which a locally normal group is isomorphically embeddable in a direct product of finite groups.

S. N. Chernikov (1) showed that a locally normal group with finite Sylow p-subgroups is isomorphically embeddable in a direct product of finite groups. P. Hall (2) removed the condition that the Sylow p-subgroups be finite, but assumed that the group is countable and embeddable in a complete direct product of finite groups (which in the preceding case was proved). As consequences of this result P. Hall obtained the following two propositions: 1) if the factor group by the center $\mathfrak{Z}$ of a locally normal group $\mathfrak{G}$ is countable, then $\mathfrak{G}/\mathfrak{Z}$ is isomorphically embeddable in a direct product of finite groups; 2) a countable locally normal group without center is isomorphic to a certain subgroup of a direct product of finite groups.

In the present paper it is shown that the first of these assertions, for groups of arbitrary cardinality, is, generally speaking, false (see the example), whereas the second is true for groups of any cardinality.

Example. Let $\mathfrak{G} = \prod_n \mathfrak{G}_n$ be the complete direct product of the groups $\mathfrak{G}_n$, $n = 1, 2, ...$, given by the generating elements A_n, B_n and the defining relations:

$$1. \quad A_n^{p^{2n}} = B_n^{p^{2n}} = 1.$$

$$2. \quad [A_n, B_n] = Z_n, \quad Z_n^{p^n} = 1.$$

$$3. \quad [A_n, Z_n] = [B_n, Z_n] = 1.$$

$$4. \quad A_n^{p^n} = B_n^{p^n} = Z_n.$$

In the group $\mathfrak{G}$ take the direct product $\mathfrak{A} = \{A_1\} \times \{A_2\} \times \cdots \times \{A_n\} \times \cdots$, and in the periodic subgroup $\mathfrak{B}$ of the complete direct product of the groups $\{B_n\}$, $n = 1, 2, \ldots$. Let the group $\mathfrak{H}$ be generated by the subgroups $\mathfrak{A}$ and $\mathfrak{B}$. Obviously, its commutator subgroup $\mathfrak{H}'$ coincides with the direct product of the groups $\{Z_n\}$, $n = 1, 2, \ldots$. It is known (see (3), p. 161) that in the abelian p-group $\mathfrak{H}'$, decomposable into a direct product of cyclic groups whose orders are not bounded in the aggregate, one can choose a subgroup $\mathfrak{K}$ such that $\overline{\mathfrak{H}} = \mathfrak{H}'/\mathfrak{K}$ is a quasicyclic group. Since the commutator subgroup $\mathfrak{H} = \mathfrak{H}\mathfrak{K}$ is quasicyclic (note that it belongs to the center $\mathfrak{Z}$ of the group $\overline{\mathfrak{H}}$), the group $\overline{\mathfrak{H}}$ is locally normal. The factor group $\overline{\mathfrak{H}}/\mathfrak{Z} \cong \mathfrak{H}/\mathfrak{Z}$, where $\mathfrak{Z}$ is the center of the group $\mathfrak{H}$, does not decompose into a direct product of cyclic groups, since it contains a subgroup isomorphic to pe-

periodic subgroup of the complete direct product of the groups $\{B_n\}/\{Z_n\}$, $n = 1, 2, \ldots$.

Thus, a locally normal group $\widetilde{\mathfrak{S}}$ has been constructed such that the factor group $\widetilde{\mathfrak{S}}/\mathfrak{Z}$ (which is countable) by its center is not embeddable in a direct product of finite groups.

Lemma 1. *If $\mathfrak{N}$ is a minimal normal divisor of a locally normal group $\mathfrak{G}$ without center, then there exists a normal divisor $\mathfrak{M}$ of finite index in $\mathfrak{G}$ intersecting $\mathfrak{N}$ in the identity, such that $\mathfrak{G}/\mathfrak{M}$ is a group without center, and $\mathfrak{N}\mathfrak{M}/\mathfrak{M}$ is the unique minimal normal divisor of the group $\mathfrak{G}/\mathfrak{M}$.*

Proof. Since the group $\mathfrak{G}$ is locally normal and has no center, for $\mathfrak{N}$ there will be found a maximal normal divisor $\mathfrak{M}$ of finite index in $\mathfrak{G}$ intersecting $\mathfrak{N}$ in the identity. Then $\mathfrak{N}\mathfrak{M}/\mathfrak{M}$ is the unique minimal normal divisor of the group $\mathfrak{G}/\mathfrak{M}$.

We shall show that $\mathfrak{G}/\mathfrak{M}$ has no center. Let $Z\mathfrak{M}$ be some element of prime order in the center of the group $\mathfrak{G}/\mathfrak{M}$. Since $[Z\mathfrak{M}, G\mathfrak{M}] = [Z, G]\mathfrak{M} = \mathfrak{M}$ for every element $G \in \mathfrak{G}$, it follows that $[G, Z] \in \mathfrak{M}$. Obviously, $Z\mathfrak{M}$ generates the group $\mathfrak{N}\mathfrak{M}/\mathfrak{M}$, and therefore as Z one may choose some element of the group $\mathfrak{N}$. But then $[G, Z] \in \mathfrak{N}$. Consequently, $\mathfrak{M} \cap \mathfrak{N} \ni [G, Z] = 1$. Thus the center of the group $\mathfrak{G}$, as well as the center of the group $\mathfrak{G}/\mathfrak{M}$, is different from the identity. The contradiction with the hypothesis of the lemma proves the absence of a center in the group $\mathfrak{G}/\mathfrak{M}$.

Lemma 2. *If, for each minimal normal divisor $\mathfrak{N}_\alpha$, $\alpha \in \mathfrak{R}$, of a locally normal group $\mathfrak{G}$ without center, in accordance with the assertion of Lemma 1 some normal divisor $\mathfrak{M}_\alpha$, $\alpha \in \mathfrak{R}$, is chosen, then under the isomorphic mapping $G \to \prod_{\alpha \in \mathfrak{R}} G\mathfrak{M}_\alpha$ of the group $\mathfrak{G}$ into the complete direct product of the groups $\mathfrak{G}/\mathfrak{M}_\alpha$, $\alpha \in \mathfrak{R}$, each element of the group $\mathfrak{G}$ is carried into an element of the group $\widetilde{\mathfrak{G}}$ that is permutable elementwise with only a finite number of the groups $\mathfrak{N}_\alpha\mathfrak{M}_\alpha/\mathfrak{M}_\alpha$.*

Proof. Let $\widetilde{G} = \prod G\mathfrak{M}_\alpha$ be the image of the element $G \in \mathfrak{G}$, and suppose that for some $\widetilde{N}_\alpha = N_\alpha \mathfrak{M}_\alpha$, $N_\alpha \in \mathfrak{N}_\alpha$, $[\widetilde{G}, \widetilde{N}_\alpha] \neq 1$, i.e. $[G, N_\alpha] \notin \mathfrak{M}_\alpha$. This means that $[G, N_\alpha] \neq 1$. Since $[G, N_\alpha] \neq [G, N_\beta]$ for $\alpha \neq \beta$, it follows from the local normality of the group $\mathfrak{G}$ that in the group $\widetilde{\mathfrak{G}}$ there exists only a finite number of subgroups $\mathfrak{N}_\alpha \mathfrak{M}_\alpha / \mathfrak{M}_\alpha$ not elementwise permutable with the given element $\widetilde{G}$.

Corollary. The image $\widetilde{G}$ of the element G under the isomorphic mapping of the group $\mathfrak{G}$ into the group $\widetilde{\mathfrak{G}}$ described in Lemma 2 has in $\widetilde{\mathfrak{G}}$ only a finite number of components distinct from the identity, belonging to normal divisors.

This follows by applying Lemma 1.

Theorem. *A locally normal group (of arbitrary cardinality) without center is isomorphically embeddable in a direct product of finite groups having no centers.*

Proof. Let $\widetilde{\mathfrak{N}}$ be the image of an arbitrary finite normal divisor of the group $\mathfrak{G}$ under the mapping of the latter into $\widetilde{\mathfrak{G}}$. In view of Lemma 1, the component $\widetilde{\mathfrak{N}}_\alpha \neq 1$ of the group $\mathfrak{N}$ in $\mathfrak{G}/\mathfrak{M}_\alpha$ intersects $\mathfrak{N}_\alpha \mathfrak{M}_\alpha / \mathfrak{M}_\alpha$ in a subgroup different from the identity, and therefore, by the finiteness of $\widetilde{\mathfrak{N}}_\alpha$, in view of the corollary of Lemma 2, it follows that the group $\widetilde{\mathfrak{N}}$ has only a finite number of such components, which proves the theorem.

Corollary 1. The factor group of a locally normal group $\mathfrak{G}$ by the union of the members of its upper central series is isomorphically embeddable in a direct product of finite groups without center.

Corollary 2 (P. Hall [2]). A countable locally normal group without center is isomorphic to some subgroup of a direct product of finite groups.

Corollary 3 (M. I. Kargapolov [4]). A locally normal group having no abelian normal divisors is isomorphically embeddable in a direct product of finite groups.

Corollary 4 (N. V. Chernikova [5]). A completely factorizable locally normal group is isomorphic to some subgroup of a direct product of finite completely factorizable groups.

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Received
15 XII 1960

References

1. S. N. Chernikov, *Matem. sborn.*, **45**, 415 (1958).

2. Ph. Hall, *J. London Math. Soc.*, **34**, No. 3, 289 (1959).

3. A. G. Kurosh, *The Theory of Groups*, Moscow, 1953.

4. M. I. Kargapolov, *Nauchn. dokl. Vyssh. shkoly, fiz.-matem. nauki*, No. 8, 3 (1958).

5. N. V. Chernikova, *Matem. sborn.*, **39**, 273 (1956).

Note: Figure translations are in progress. See original paper for figures.