THE STRUCTURE OF THE RESOLVENT OF THE SCHRÖDINGER OPERATOR FOR A SYSTEM OF THREE PARTICLES WITH PAIR INTERACTION

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Abstract

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THE STRUCTURE OF THE RESOLVENT OF THE SCHRÖDINGER OPERATOR FOR A SYSTEM OF THREE PARTICLES WITH PAIR INTERACTION

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The energy operator of a system of N pairwise interacting particles with masses $m_1, \dots, m_N$ has the form

$$H_N = -\sum_{i=1}^{N} \frac{1}{2m_i} \nabla_i^2 + \sum_{i<j}^{N} v_{ij}(r_i - r_j). \tag{1}$$

Here r_i is the radius vector of the i-th particle and ∇_i^2 is the three-dimensional Laplace operator with respect to the variable r_i. In scattering theory all potentials $v_{ij}(r)$ decrease when $|r| \to \infty$.

At the present time only the operator H_2 has been studied in detail. Namely, A. Ya. Povzner ([1,2]) studied the spectrum of this operator under certain conditions on $v_{12}(r)$ and proved an expansion theorem in its eigenfunctions. These results were refined by Kato ([3]) and Ikebe ([4]). The basis of A. Ya. Povzner's approach is the study of the resolvent of the operator H_2 in the complex plane and especially in a neighborhood of the real axis. The method used by him does not carry over directly to the case $N > 2$.

Recently the author proposed, in a physical paper ([5]), new integral equations for the investigation of a system with three particles. In the present work we give some results, obtained with the aid of these equations, concerning the behavior of the resolvent of the operator H_3 in the complex plane. The main result is formulated in the theorem.

1. Instead of the operator H_3 it is more convenient for us to study the operator H, obtained from H_3 after passing to the momentum representation (i.e., after a Fourier transform) and separating off the energy operator for the motion of the center of inertia. For lack of space we do not describe the corresponding transition. To simplify the formulae, we assume that $m_1 = m_2 = m_3$. The operator H is described below.

Consider three vectors p_1, p_2, and p_3, connected by the relation

$$p_1 + p_2 + p_3 = 0, \tag{2}$$

so that each pair p_1, p_2; p_2, p_3 and p_3, p_1 independently ranges over the whole six-dimensional space E_6. The operator H is defined in $\mathcal{L}_2(E_6)$ and has the form

$$H = H_0 + V = H_0 + V_{23} + V_{31} + V_{12}, \tag{3}$$

where H_0 is the operator of multiplication by the function

$$p_1^2 + (p_1 p_2) + p_2^2 = p_2^2 + (p_2 p_3) + p_3^2 = p_3^2 + (p_3 p_1) + p_1^2 \tag{4}$$

with its natural domain of definition; the operator V_{23} has the kernel

$$V_{23}(p, p') = v_{23}(p_2 - p_2')\, \delta(p_1 - p_1'), \tag{5}$$

and the operators V_{31} and V_{12} are defined in an analogous way in the coordinates p_3, p_2 and p_1, p_3, respectively.

For the functions $v_{ij}(q) = \overline{v_{ij}(-q)}$ we require the estimate

$$|v_{ij}(q)| \leq C(1 + |q|)^{-1-\varepsilon_0}, \qquad \varepsilon_0 > 0, \qquad i, j = 1, 2, 3. \tag{6}$$

Under this condition the operators $(H_0 + I)^{-\alpha} V_{ij} (H_0 + I)^{-\alpha}$, for $\alpha > 1/2 - \varepsilon_0/4$, are bounded in $\mathcal{L}_2(E_6)$, which makes it possible to define uniquely, by means of the quadratic form, the self-adjoint operator H.

2. Let $R(z) = (H - zI)^{-1}$ and $R_0(z) = (H_0 - zI)^{-1}$ be the resolvents of the operators H and H_0, respectively. The relation

$$R(z) = R_0(z) - R_0(z) V R(z) \tag{7}$$

holds; moreover, if one seeks $R(z)$ in the form

$$R(z) = R_0(z) - R_0(z) T(z) R_0(z), \tag{8}$$

then for $T(z)$ the equivalent relation has the form

$$T(z) = V - V R_0(z) T(z). \tag{9}$$

Equations of type (7) or (9) have proved useful in the study of the operator H_2, and in the momentum representation equation of type (9) is more convenient.

For application to our problem, equation (9) should be "partially solved." This partial solution is effected by means of the following device, which seems somewhat artificial, but in fact has a quite natural basis, which we cannot give for lack of space.

Consider the matrix equation (here and below Gothic letters denote 3×3 matrices composed of operators)

$$\mathfrak{T}(z) = \begin{pmatrix} V_{23} & 0 & 0 \\ 0 & V_{31} & 0 \\ 0 & 0 & V_{12} \end{pmatrix} - \begin{pmatrix} V_{23} & V_{23} & V_{23} \\ V_{31} & V_{31} & V_{31} \\ V_{12} & V_{12} & V_{12} \end{pmatrix} R_0(z)\mathfrak{T}(z). \tag{10}$$

This equation is connected with equation (9) in the sense that the sum of all matrix elements of the matrix $\mathfrak{T}(z)$ satisfies equation (9). Let $T_{23}(z)$ be the solution of the equation

$$T_{23}(z) = V_{23} - V_{23}R_0(z)T_{23}(z), \tag{11}$$

and let $T_{31}(z)$ and $T_{12}(z)$ be defined analogously. Inverting the diagonal part in equation (10), we arrive at the equation

$$\mathfrak{T}(z) = \begin{pmatrix} T_{23}(z) & 0 & 0 \\ 0 & T_{31}(z) & 0 \\ 0 & 0 & T_{12}(z) \end{pmatrix} - \begin{pmatrix} 0 & T_{23}(z) & T_{23}(z) \\ T_{31}(z) & 0 & T_{31}(z) \\ T_{12}(z) & T_{12}(z) & 0 \end{pmatrix} R_0(z)\mathfrak{T}(z) = \mathfrak{T}_0(z) - \mathfrak{A}(z)\mathfrak{T}(z). \tag{12}$$

Using equation (11) one can verify that the operator $T_{23}(z)$ has the kernel

$$T_{23}(p, p') = t_{23}\left(-p_2 - \frac{1}{2}p_1,\, -p_2' - \frac{1}{2}p_1,\, z - \frac{3}{4}p_1^2\right)\delta(p_1 - p_1'), \tag{13}$$

where $t_{23}(k, k', z)$ is the solution of an integral equation of the type

$$t(k, k', z) = v(k - k') - \int v(k - k'')\,(k''^2 - z)^{-1}\, t(k'', k', z)\, dk''. \tag{14}$$

In order not to have δ-type singularities in the free term, it is more convenient to study the matrix $\mathfrak{W}(z) = \mathfrak{T}(z) - \mathfrak{T}_0(z)$, the equation for which has the form

$$\mathfrak{W}(z) = \mathfrak{W}_0(z) - \mathfrak{A}(z)\mathfrak{W}(z), \tag{15}$$

where

$$\mathfrak{W}_0(z) = \begin{pmatrix} 0 & T_{23}(z)R_0(z)T_{31}(z) & T_{23}(z)R_0(z)T_{12}(z) \\ T_{31}(z)R_0(z)T_{23}(z) & 0 & T_{31}(z)R_0(z)T_{12}(z) \\ T_{12}(z)R_0(z)T_{23}(z) & T_{12}(z)R_0(z)T_{31}(z) & 0 \end{pmatrix}. \tag{16}$$

With the aid of equation (15) it proved possible to study the behavior of the resolvent $R(z)$ for $\operatorname{Im} z \neq 0$.

3. On the basis of the study of equation (14), under condition (6) one can obtain that, for $t'_{ij}(k, k', z)$, the estimate

$$|t'_{ij}(k, k', z)| \leq C(1 + |k - k'|)^{-1-\varepsilon}, \quad 0 < \varepsilon < \varepsilon_0, \quad i, j = 1, 2, 3, \tag{17}$$

holds uniformly in z in the domain

$$\delta \leq \arg z \leq 2\pi - \delta, \quad |\operatorname{Im} z| \geq \delta, \quad \delta > 0. \tag{18}$$

For writing estimates in the study of equation (15) it is convenient to use the functions

$$M_{ij}(p, p', \varepsilon) = (p_i^2 + p_{j'}^2 + 1)^{-1} \sum_{k<l}^{3} \left[(1 + |p_k - p'_k|)(1 + |p_l - p'_l|)\right]^{-1-\varepsilon}. \tag{19}$$

With the aid of estimates (17) one can obtain, for example, that

$$|(T_{23}(z)R_0(z)T_{31}(z))(p, p')| \leq CM_{12}(p, p', \varepsilon). \tag{20}$$

It is natural to study equation (15) in the class of matrices $\mathfrak{W}(z)$ for whose elements the estimate

$$|W_{ij}(p, p')| \leq CM_{ij}(p, p', \varepsilon), \quad i, j = 1, 2, 3 \tag{21}$$

holds.

It is not difficult to construct the corresponding Banach space $B(\varepsilon)$. The results of the investigation of equation (15) may be formulated as follows:

Let z vary in the domain (18). Then the following assertions hold:

I. The operator $\mathfrak{A}(z)$ is bounded in $B(\varepsilon)$.

II. The homogeneous equation $\varphi + \mathfrak{A}(z)\varphi = 0$ has no nontrivial solutions in $B(\varepsilon)$.

III. The operator $\mathfrak{A}^2(z)$ is completely continuous in $B(\varepsilon')$ for any $\varepsilon' < \varepsilon$.

On the basis of a theorem of S. M. Nikol' skii[6], it follows from assertion III that the Fredholm alternative is applicable to the operator $\mathfrak{A}(z)$, and from assertion II it follows that equation (15) is uniquely solvable in $B(\varepsilon')$.

Thus we arrive at the following main result:

Theorem. *Let the potentials V_{ij} satisfy condition (6). Then the resolvent of the operator H has the following structure:*

$$R(z) = R_0(z) + \sum_{i<j}^{3}(R_{ij}(z) - R_0(z)) + R_0(z)W(z)R_0(z). \tag{22}$$

Here $R_0(z)$ and $R_{ij}(z)$ are the resolvents of the operators H_0 and $H_{ij} = H_0 + V_{ij}$, $i, j = 1, 2, 3$, respectively, and $W(z)$ is an integral operator whose kernel satisfies the estimate

$$|W(p, p')| \leq C \sum_{i,j=1}^{3} M_{ij}(p, p', \varepsilon), \tag{23}$$

where $\varepsilon < \varepsilon_0$ in estimate (6), uniformly in z in the domain (18).

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Note: Figure translations are in progress. See original paper for figures.