# ON A PROBLEM OF I. M. GELFAND

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## Abstract

## Full Text

## MATHEMATICS

**A. A. Kirillov**

# ON A PROBLEM OF I. M. GELFAND

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Let a curve $K$ be given in an $n$-dimensional (real or complex) space. The totality of all lines intersecting $K$ forms an $n$-dimensional (real or complex) manifold $M$. To each rapidly decreasing infinitely differentiable function $f(x)$ in the original space one can associate a function $\hat{f}(m)$ on the manifold $M$: $\hat{f}(m)$ is equal to the integral of $f(x)$ over the line $m \in M$. We shall show that the function $\hat{f}(x)$ can be reconstructed from $f(m)$ if and only if the curve $K$ intersects almost all hyperplanes $(x, \xi) = c$. In the case where the multiplicity of intersection is the same for almost all hyperplanes, it turns out to be possible to give an explicit formula expressing $f(x)$ in terms of $\hat{f}(m)$. The result obtained gives an answer to one of the questions posed by I. M. Gelfand in ([1]).

In what follows we shall restrict ourselves to the case of the complex space $C_n$ (for a real space the same formulas hold with obvious changes). Let the curve $K$ be given parametrically: $x = \varphi(\lambda)$, where $\lambda$ is a complex parameter. Denote by $g(a, \lambda)$ the integral of the function $f(x)$ over the line passing through the point $\varphi(\lambda)$ with direction vector $a$:

$$g(a, \lambda) = \int f(\varphi(\lambda) + ta)\, dt\, d\bar{t}. \tag{1}$$

Our problem is to reconstruct the function $f(x)$, knowing $g(a, \lambda)$. Denote by $G(\beta, \lambda)$ the Fourier transform of the function $g(a, \lambda)$ with respect to the variables $a^*$. Then

$$\begin{aligned} G(\beta, \lambda) &= \int g(a, \lambda) e^{i\,\mathrm{Re}(a,\beta)}\, da\, d\bar{a} \\ &= \iiint f(\varphi(\lambda) + ta) e^{i\,\mathrm{Re}(a,\beta)}\, dt\, d\bar{t}\, da\, d\bar{a} \\ &= \iint f(\gamma) e^{i\,\mathrm{Re}\left(\frac{\gamma-\varphi(\lambda)}{t},\beta\right)} |t|^{-2n}\, d\gamma\, d\bar{\gamma}\, dt\, d\bar{t} \\ &= \iint f(\gamma) e^{i\,\mathrm{Re}(\gamma-\varphi(\lambda),\tau\beta)} |\tau|^{2n-4}\, d\gamma\, d\bar{\gamma}\, d\tau\, d\bar{\tau} \\ &= \int \tilde{f}(\tau\beta) |\tau|^{2n-4} e^{-i\,\mathrm{Re}(\varphi(\lambda),\tau\beta)}\, d\tau\, d\bar{\tau}, \end{aligned}$$

where $\tilde{f}(\beta)$ is the Fourier transform of the function $f(x)$. Introduce the functions

$$\Phi(\beta,\tau)=\tilde{f}(\tau\beta)|\tau|^{2n-4} \quad \text{and} \quad F(\beta,\omega)=\int \Phi(\beta,\tau)e^{-i\operatorname{Re}\omega\tau}\,d\tau\,d\bar{\tau}^{**}.$$

The computations given above show that

$$G(\beta,\lambda)=F(\beta,(\varphi(\lambda),\beta)). \tag{2}$$

It is obvious that the functions $f$ and $F$, as well as the functions $g$ and $G$, are connected by invertible transformations. Therefore it remains to investigate the relation between $F$ and $G$. Relation (2) shows that, knowing the function $G$, one can deter-

* The function $g(a,\lambda)$ is homogeneous: $g(ta,\lambda)=|t|^{-2}g(a,\lambda)$. Therefore its Fourier transform is a generalized homogeneous function (see, for example, ([2]), Chapter III).

** It is easy to see that for $\beta\neq 0$ this integral converges, since $\tilde{f}$ is a rapidly decreasing function.

to determine the value $F(\beta,\omega)$ only when, for some $\lambda$, the equality $(\varphi(\lambda),\beta)=\omega$ holds. This condition means that the curve $K$ intersects the hyperplane $(x,\beta)=\omega$.

Thus, in order that the function $F$ be recoverable from the function $G$ (and hence the function $f$ from $g$), it is necessary and sufficient that the curve $K$ intersect almost all hyperplanes. For the case of a plane curve in three-dimensional real space this result was obtained earlier by I. Ya. Vakhutinskii.

Now suppose that the curve $K$ intersects almost all hyperplanes in exactly $l$ points. This assumption is satisfied, for example, when $K$ is an algebraic curve. For almost every $\beta$ one can divide the domain traversed by the parameter $\lambda$ into $l$ parts $\Lambda_1,\dots,\Lambda_l$ in such a way that, for $\lambda\in\Lambda_i$, the quantity $\omega=(\varphi(\lambda),\beta)$ traverses almost the entire complex plane. We now use the expression of $f$ in terms of $F$:

$$f(x)=(2\pi)^{-2n-2}\iint F(\beta,\omega)e^{i\operatorname{Re}[\omega-(\beta,x)]}\,d\beta\,d\bar{\beta}\,d\omega\,d\bar{\omega}$$

and substitute into it the expression of $F$ in terms of $G$ from (2). We obtain:

$$
\begin{aligned}
f(x) &= (2\pi)^{-2n-2} \iint_{\lambda\in\Lambda_i} G(\beta,\lambda) e^{i\,\mathrm{Re}(\varphi(\lambda)-x,\beta)} \frac{D(\omega,\bar\omega)}{D(\lambda,\bar\lambda)}\, d\beta\, d\bar\beta\, d\lambda\, d\bar\lambda \\
&= -\frac{1}{l}(2\pi)^{-2n-2} \iiint g(\alpha,\lambda) e^{i\,\mathrm{Re}(\varphi(\lambda)-x+\alpha,\beta)} \left( \left|\frac{\partial\omega}{\partial\lambda}\right|^2 - \left|\frac{\partial\omega}{\partial\bar\lambda}\right|^2 \right) d\alpha\, d\bar\alpha\, d\beta\, d\bar\beta\, d\lambda\, d\bar\lambda \\
&= -\frac{1}{4l\pi^2} \int Dg(x-\varphi(\lambda),\lambda)\, d\lambda\, d\bar\lambda,
\end{aligned}
$$

where

$$
Dg(\alpha,\lambda) = \sum \left( \frac{\partial\varphi_i}{\partial\lambda}\frac{\partial\varphi_j}{\partial\bar\lambda} - \frac{\partial\varphi_i}{\partial\bar\lambda}\frac{\partial\varphi_j}{\partial\lambda} \right) \frac{\partial^2 g(\alpha,\lambda)}{\partial\alpha_i\,\partial\bar\alpha_j}.
$$

In the case of an algebraic curve, $\varphi$ depends analytically on $\lambda$. Therefore $\partial\varphi_i/\partial\bar\lambda = 0$, and for the operator $D$ we obtain the simpler expression

$$
D = \sum \frac{\partial\varphi_i}{\partial\lambda}\frac{\partial\bar\varphi_j}{\partial\bar\lambda}\frac{\partial^2}{\partial\alpha_i\,\partial\bar\alpha_j}.
$$

The formula we have obtained

$$
f(x) = -\frac{1}{4l\pi^2} \int Dg(x-\varphi(\lambda),\lambda)\, d\lambda\, d\bar\lambda, \tag{3}
$$

admits a simple geometric interpretation: to recover the value $f(x)$, it is necessary to know the integrals of the function $f$ over the lines passing through the point $x$ (and intersecting $K$) and over lines close to them.

Formula (3) can be rewritten so that the parametric specification of the curve does not enter it. To this end, note that on the curve $x = \varphi(\lambda)$ the differential form

$$
\frac{\partial\varphi_i}{\partial\lambda}\frac{\partial\bar\varphi_j}{\partial\bar\lambda}\, d\lambda\, d\bar\lambda
$$

coincides with $dx_i\, d\bar x_j$. Therefore, denoting by $h(a,x)$ the integral of the function $f$ over the line passing through the point $x$ with direction vector $a$, we obtain

$$
f(x_0) = -\frac{1}{4l\pi^2} \int_K \sum \frac{\partial^2 h(x_0-x,x)}{\partial a_i\,\partial\bar a_j}\, d\bar x_i\, d\bar x_j. \tag{4}
$$

For the case when $K$ is the "hyperbola" in $C^3$ given by the equations $z_1 z_2 = 1$, $z_3 = 0$, formula (4) was obtained by I. M. Gelfand (1).

Moscow State University
named after M. V. Lomonosov

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## REFERENCES

1. I. M. Gelfand, *UMN*, No. 2, 161 (1960).

2. I. M. Gelfand, G. E. Shilov, *Generalized Functions and Operations on Them*, Moscow, 1959.

*Note: Figure translations are in progress. See original paper for figures.*