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Abstract

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ASYMPTOTICS OF THE GREEN' S FUNCTION OF AN ORDINARY LINEAR DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIENTS DEPENDING ON A SMALL PARAMETER

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Let

$$L_\varepsilon = \sum_{s=1}^{l} \varepsilon^s a_{k+s}(x) D_x^{k+s} + L_0, \qquad L_0 = \sum_{j=0}^{k} a_j(x) D_x^j, \tag{1}$$

where D_x^j denotes j-fold differentiation with respect to x. The coefficients $a_q(x)$ $(0 \le q \le k+l)$ are assumed to be sufficiently smooth on the interval $[x_1, x_2]$, and $a_{k+l}(x)$ and $a_k(x)$ are nowhere zero. Suppose that, for certain k_1, k_2 $(k_1 + k_2 = k)$, the problem

$$L_0 u = 0, \qquad D_x^i u\big|_{x_m} = 0 \quad (0 \le i \le k_m - 1;\ m = 1, 2) \tag{2}$$

has only the zero solution. Suppose also that the following condition is fulfilled:

(R) The multiplicity of the roots of the algebraic equation

$$Q(w) \equiv \sum_{s=0}^{l} a_{k+s}(x) \omega^s = 0 \tag{3}$$

does not depend on $x \in [x_1, x_2]$.

In addition, let (3) have no roots on the imaginary axis. Then in the left half-plane there is a certain number l_1 (counting multiplicities) of these roots: $w_1(x), \dots, w_{l_1}(x)$, and in the right half-plane $l_2 = l - l_1$ roots: $w_{l_1+1}(x), \dots, w_l(x)$.

These conditions are obviously sufficient for, when $\varepsilon < \varepsilon_0$, the boundary-value problem

$$L_\varepsilon u = 0, \qquad D_x^i u\big|_{x_m} = 0, \qquad D_x^{k_m+p} u\big|_{x_m} = 0$$

$$(0 \le i \le k_m - 1; \qquad 0 \le p \le l_m - 1)$$

to likewise have only the zero solution.* Hence it follows that there exist corresponding Green’ s functions $G_0(x,\xi)$ and $G_\varepsilon(x,\xi)$, defined as solutions of the following problems (in x):

$$L_0 G_0 = 0, \qquad D_x^i G_0\big|_{x_m} = 0, \qquad D_x^j G_0\big|_{\xi-0}^{\xi+0} = \delta_{j,k-1} a_k^{-1}(\xi); \tag{4}$$

$$L_\varepsilon G_\varepsilon = 0, \qquad D_x^i G_\varepsilon\big|_{x_m} = 0, \qquad D_x^{k_m+p} G_\varepsilon\big|_{x_m} = 0, \qquad D_x^j G_\varepsilon\big|_{\xi-0}^{\xi+0} = 0, \tag{5}$$

$$D_x^{k+r} G_\varepsilon\big|_{\xi-0}^{\xi+0} = \delta_{r,l-1} \varepsilon^{-s} a_{k+l}^{-1}(\xi),$$

where $f(x)\big|_{\xi-0}^{\xi+0}$ denotes the jump $f(\xi+0) - f(\xi-0)$. Here and below: $0 \le i \le k_m - 1$, $\quad 0 \le j \le k-1$, $\quad 0 \le p \le l_m - 1$, $\quad 0 \le r \le l-1$, $m = 1, 2$. As $\varepsilon \to 0$, G_ε cannot converge to G_0 uniformly with $k+l$ derivatives on the entire interval $[x_1, x_2]$. (However, we shall establish below that such convergence takes place on any closed set containing none of the points x_1, x_2, ξ.)

In the present note, the behavior of G_ε for small ε in neighborhoods of the points x_1, x_2, ξ is considered in detail; an approximate formula for G_ε is constructed and a certain estimate of the remainder term is given.

* This was established by A. B. Shabat and also follows independently from Lemma 1.

We shall give the definition due to M. I. Vishik and L. A. Lyusternik in (^{1}). We shall say that a q-times continuously differentiable function $v_\varepsilon(x)$ is a function of boundary-layer type of order s in a neighborhood of the point $x = c$ $(s \le q)$, if, as $\varepsilon \to 0$, v_ε together with q derivatives tends uniformly to zero on every closed set not containing the point c; while in the whole neighborhood of the point c the derivatives of v_ε up to order $s-1$ tend to zero, the s-th derivative is bounded, and the $(s+1)$-st derivative tends to $\pm\infty$. We shall prove that $G_\varepsilon - G_0$ in neighborhoods of the points x_1, x_2, ξ is a function of boundary-layer type of orders $k_1, k_2, k-1$, respectively.

Denote $a_{k+s}(x) D_x^{k+s} = L_s$; then $L_\varepsilon = L_0 + \varepsilon L_1 + \dots + \varepsilon^l L_l$. Introduce $t = (x-c)\varepsilon^{-1}$, where c is one of the three points: x_1, x_2, ξ. Expanding the coefficients of the equation by Taylor’ s formula and noting that $D_x = \varepsilon^{-1} D_t$, we have

$$L_\varepsilon = \varepsilon^{-k} \left(M_0 + \varepsilon M_1 + \dots + \varepsilon^N M_N + \varepsilon^{N+1} \widetilde{M}_{N+1} \right),$$

$$M_0 = a_{k+l}(c)D_t^{k+l} + a_{k+l-1}(c)D_t^{k+l-1} + \cdots + a_k(c)D_t^k;$$

M_s are linear differential operators with bounded coefficients, depending polynomially on t for $s \leq N$. We shall seek G_ε in the form:

$$G_\varepsilon(x,\xi) = u_{\varepsilon,n}(x,\xi) - \varepsilon^{k_1}v_{\varepsilon,1,n} + \varepsilon^{k_2}v_{\varepsilon,2,n} + \varepsilon^{k-1}v_{\varepsilon,n} + z_n, \tag{6}$$

where z_n is the remainder term, which we shall estimate below. Here

$$u_{\varepsilon,n} = u_0 + \varepsilon u_1 + \cdots, \qquad v_{\varepsilon,m,n} = v_{m,0} + \varepsilon v_{m,1} + \cdots, \qquad v_{\varepsilon,n} = v_0 + \varepsilon v_1 + \cdots$$

(the sums in powers of ε are finite; the number of terms in each will be specified later). We shall show that $u_0 = G_0$; $v_{m,s}$, v_s, as functions of x, are functions of boundary-layer type of zero order in neighborhoods of the points x_m, ξ, respectively, and

$$v_{m,s} = v_{m,s}(t,\xi) \quad (t = (x - x_m)\varepsilon^{-1}); \qquad v_s = v_s(t,\xi) \quad (t = (x-\xi)\varepsilon^{-1}).$$

Further we proceed formally, following the method developed in (^{1}). Substitute (6) into (5). When substituting (6) into the first of conditions (5), we set equal to zero the operator L_ε applied to each of the first four terms in the sum (6). Then we obtain:

$$L_\varepsilon u_{\varepsilon,n} = (L_0 + \varepsilon L_1 + \cdots)(u_0 + \varepsilon u_1 + \cdots) = 0; \tag{7}$$

$$L_\varepsilon \varepsilon^{k_m} v_{\varepsilon,m,n} = \varepsilon^{k_m - k}(M_0 + \varepsilon M_1 + \cdots)(v_{m,0} + \varepsilon v_{m,1} + \cdots) = 0; \tag{8}$$

$$L_\varepsilon \varepsilon^{k-1} v_{\varepsilon,n} = \varepsilon^{-1}(M_0 + \varepsilon M_1 + \cdots)(v_0 + \varepsilon v_1 + \cdots) = 0. \tag{9}$$

Substituting (6) into the boundary conditions, we take into account that $v_{m,n}$ and v_n are functions of boundary-layer type in neighborhoods of the points x_m and ξ, respectively. Then we have:

$$D_x^i(u_0 + \varepsilon u_1 + \cdots)\big|_{x_m} + \varepsilon^{k_m - i}D_t^i(v_{m,0} + \varepsilon v_{m,1} + \cdots)\big|_0 = 0; \tag{10}$$

$$D_x^{k_m+p}(u_0 + \varepsilon u_1 + \cdots)\big|_{x_m} + \varepsilon^{-p}D_t^{k_m+p}(v_{m,0} + \varepsilon v_{m,1} + \cdots)\big|_0 = 0; \tag{11}$$

$$D_x^i(u_0 + \varepsilon u_1 + \cdots)\big|_{\xi-0}^{\xi+0} + \varepsilon^{k-1-i}D_t^i(v_0 + \varepsilon v_1 + \cdots)\big|_{-0}^{+0} = 0; \tag{12}$$

$$D_x^{k+r}(u_0 + \varepsilon u_1 + \cdots)\big|_{\xi-0}^{\xi+0} + \varepsilon^{-1-r}D_t^{k+r}(v_0 + \varepsilon v_1 + \cdots)\big|_{-0}^{+0} = \delta_{r,l-1}\varepsilon^{-l}a_{k+l}^{-1}(\xi). \tag{13}$$

Equating the terms with identical powers of ε in the purely formal equalities (7)–(13), we obtain a recurrent system for the functions $v_0, u_0, v_{1,0}, v_{2,0}, \ldots, v_s, u_s, v_{1,s}, v_{2,s}, \ldots$.

Equating in (9) the terms with ε^{-1}, and in (13) the terms with ε^{-1-r}, we obtain:

$$M_0 v_0 = 0, \qquad D_t^{k+r}v_0\big|_{-0}^{+0} = \delta_{r,l-1}a_{k+l}^{-1}(\xi). \tag{14}$$

Further, to simplify the exposition, we assume that the roots of equation (3) are simple—

although the main theorem is also valid in the general case. We seek v_0 in the form:

$$v_0 = \sum_{q=1}^{l_1} c_q(\xi) \exp w_q(\xi) t \quad (t > 0);$$

$$v_0 = \sum_{q=1}^{l_2} c_{q+l_1}(\xi) \exp w_{q+l_1}(\xi) t \quad (t < 0). \tag{15}$$

The c_q are determined from a system of linear equations with determinant $W \neq 0$. Equating in (7), (10), (12) the terms with ε^0, we obtain:

$$L_0 u_0 = 0, \qquad D_x^i u_0\big|_{x_m} = 0, \qquad D_x^j u_0\big|_{\xi-0}^{\xi+0} = -\delta_{j,k-1} D_t^{k-1} v_0\big|_{-0}^{+0}. \tag{16}$$

It is easy to see that $-D_t^{k-1} v_0\big|_{-0}^{+0} = a_k^{-1}(\xi)$; comparing (4) and (16), we find that $u_0 = G_0$. Equating in (8) the terms with ε^{km-k}, and in (11) those with ε^{-p}, we have the system (solvable as in (1)):

$$M_0 v_{m,0} = 0, \qquad D_t^{km+p} v_{m,0}\big|_0 = -\delta_{0,p} D_x^{km+p} u_0\big|_{x_m}. \tag{17}$$

If $v_0, u_0, v_{1,0}, v_{2,0}, \dots, v_{s-1}, u_{s-1}, v_{1,s-1}, v_{2,s-1}$ $(s \geq 1)$ have already been found, then the equations for determining $v_s, u_s, v_{1,s}, v_{2,s}$ are obtained as follows. The system for v_s is found by equating the terms with ε^{-1+s} in (9) and the terms with ε^{-1-r+s} in (13):

$$M_0 v_s = -\sum_{q=0}^{s-1} M_{s-q} v_q, \qquad D_t^{k+r} v_s\big|_{-0}^{+0} = -D_x^{k+r} u_{s-r-1}\big|_{-0}^{+0}. \tag{18}$$

The system for u_s is obtained from (7), (10), and (12) by equating the terms with ε^s:

$$L_0 u_s = -\sum_{q=0}^{s-1} L_{s-q} u_q, \quad D_x^i u_s\big|_{x_m} = -D_t^i v_{s-km+i|0}, \quad D_x^j u_s\big|_{-0}^{+0} = -D_t^j v_{s-k+j}\big|_{-0}^{+0}. \tag{19}$$

The systems for $v_{m,s}$ are obtained from (8) and (11), when we equate the terms with ε^{km-k+s} and ε^{-p+s}, respectively:

$$M_0 v_{m,s} = -\sum_{q=0}^{s-1} M_{s-q} v_{m,q}, \qquad D_t^{km+p} v_s\big|_0 = -D_x^{km+p} u_{s-q}\big|_{x_m}. \tag{20}$$

Here, for compactness of notation, it is denoted that $L_q = 0$ $(q > l)$; $v_q = 0$, $u_q = 0$, $v_{m,q} = 0$ $(q < 0)$. Simple inductive considerations show the possibility of successive solution of the system (18)–(20). In this case $u_s(x, \xi)$ has $k + l$ continuous derivatives $(x \neq \xi)$ and does not depend on ε. The functions $v_s, v_{1,s}, v_{2,s}$ have the form:

$$v_{1,s} = \sum_{q=1}^{l_1} Q_{1,s,q}(t) \exp w_q(x_1)t, \qquad t = (x - x_1)\varepsilon^{-1} > 0; \tag{21}$$

$$v_{2,s} = \sum_{q=1}^{l_2} Q_{2,s,l_1+q}(t) \exp w_{l_1+q}(x_2)t, \qquad t = (x - x_2)\varepsilon^{-1} < 0; \tag{22}$$

$$v_s = \sum_{q=1}^{l_1} Q_{s,q}(t) \exp w_q(\xi)t, \qquad t = (x - \xi)\varepsilon^{-1} > 0; \tag{23}$$

$$v_s = \sum_{q=1}^{l_2} Q_{s,l_1+q}(t) \exp w_{l_1+q}(\xi)t, \qquad t = (x - \xi)\varepsilon^{-1} < 0. \tag{24}$$

where $Q_{1,s,q}, Q_{2,s,q}, Q_{s,q}, Q_{s,l_1+q}$ are polynomials in t with coefficients depending on ξ and independent of ε.

Let us now set the n-th approximation to G_ε equal to

$$G_n = u_{\varepsilon,n} + \varepsilon^{k_1} v_{\varepsilon,1,n} + \varepsilon^{k_2} v_{\varepsilon,2,n} + \varepsilon^{k-1} v_{\varepsilon,n},$$

where

$$u_{\varepsilon,n} \equiv G_0 + \varepsilon u_1 + \dots + \varepsilon^n u_n, \qquad v_{\varepsilon,m,n} \equiv v_{m,0} + \varepsilon v_{m,1} + \dots + \varepsilon^{n+k-k_m} v_{m,n+k-k_m},$$

$$v_{\varepsilon,n} \equiv v_0 + \varepsilon v_1 + \dots + \varepsilon^{n+1} v_{n+1}.$$

To estimate the remainder $z_n \equiv G_\varepsilon - G_n$, we determine with what accuracy G_n satisfies conditions (5). It is easily calculated that

$$L_\varepsilon z_n = O(\varepsilon^{n+1}), \qquad D_x^i z_n\big|_{x_m} = O(\varepsilon^{n+1}), \qquad D_x^{k_m+p} z_n\big|_{x_m} = O(\varepsilon^{n+1-p}),$$

$$D_x^j z_n\big|_{\xi=-0}^{\xi=+1} = O(\varepsilon^{n+1}), \qquad D_x^{k+r} z_n\big|_{\xi=-0}^{\xi=+0} = O(\varepsilon^{n+1-r}). \tag{25}$$

Next, the estimate of z_n is carried out with the aid of the lemma:

Lemma 1. If $\varepsilon < \varepsilon_0$, there exists, and is unique, the solution of the problem

$$L_\varepsilon u = h, \qquad D_x^i u\big|_{x_m} = d_{m,i}, \qquad D_x^{k_m+p} u\big|_{x_m} = \varepsilon^{-p} d_{m,k_m+p},$$

$$D_x^j u\big|_{\xi=-0}^{\xi=+0} = d_j, \qquad D_x^{k+r} u\big|_{\xi=-0}^{\xi=+0} = \varepsilon^{-r} d_{k+r},$$

where h is continuous for $x \neq \xi$. Moreover, if for arbitrary $\delta > 0$ we define

$$F_m = \{x : |x - x_m| < \delta\} \cap \{x : x_1 \le x \le x_2\}, \qquad F_\xi = \{x : |x - \xi| < \delta\} \cap \{x : x_1 \le x \le x_2\},$$

$$H = \{x : x_1 \le x \le x_2\} \setminus (F_1 \cup F_2 \cup F_\xi),$$

$$\|u\|_{\varepsilon,\delta} \equiv \sum_{s=0}^{k+l} \sup_{x \in H} |D_x^s u| + \sum_{m=1}^{2} \left(\sum_{i=0}^{k_m-1} \sup_{x \in F_m} |D_x^i u| + \sum_{p=0}^{p=k+l-k_m} \varepsilon^p \sup_{x \in F_m} |D_x^{k_m+p} u| \right) + \\ + \sum_{j=0}^{k-1} \sup_{x \in F_\xi} |D_x^j u| + \sum_{r=0}^{r+l} \varepsilon^r \sup_{x \in F_\xi} |D_x^{k+r} u|, \tag{26}$$

then the estimate holds:

$$\|u\|_{\varepsilon,\delta} \le K \left(\sup_{x \in [x_1, x_2]} |h| + \sum_{m=1}^{2} \sum_{s=0}^{k_m + l_m - 1} |d_{m,s}| + \sum_{s=0}^{k+l-1} |d_s| \right),$$

where K is bounded for $\delta > \delta_0 > 0,\ |\xi - x_m| > \eta_0 > 0,\ \varepsilon < \varepsilon_0$.

In the proof of the lemma one uses theorem (^{2}) on the form of a fundamental system of solutions of the equation $L_\varepsilon z = 0$.

From (26), on the basis of Lemma 1, it follows that

$$\|z_n\|_{\varepsilon,\delta} = O(\varepsilon^{n+1}).$$

Thus we arrive at the theorem:

Theorem. If the Green' s function $G_0(x, \xi)$ exists and condition (R) is satisfied, then for $\varepsilon < \varepsilon_0$ there also exists $G_\varepsilon(x, \xi)$, and the following representation holds:

$$G_\varepsilon = G_0 + \sum_{s=1}^{n} \varepsilon^s u_s + \varepsilon^{k_1} \sum_{s=0}^{n+k_2} \varepsilon^s v_{1,s} + \varepsilon^{k_2} \sum_{s=0}^{n+k_1} \varepsilon^s v_{2,s} + \varepsilon^{k-1} \sum_{s=0}^{n+1} \varepsilon^s v_s + z_n. \tag{27}$$

Here $u_s, v_{1,s}, v_{2,s}, v_s$ are determined from the system (14), (16), (17), (19)–(21). The functions $u_s(x, \xi)$ have $k + l$ derivatives with respect to x for $x \neq \xi$ and do not depend on ε. $v_{1,s}, v_{2,s}, v_s$ are functions of boundary-layer type of zero order in neighborhoods of the points x_1, x_2, ξ, respectively, and have the form (21)–(24). $\|z_n\|_{\varepsilon,\delta} \le C\varepsilon^{n+1}$, where $\|\ \|_{\varepsilon,\delta}$ is defined by expression (26); C is bounded if $\delta > \delta_0 > 0,\ |\xi - x_m| > \eta_0 > 0,\ \varepsilon < \varepsilon_0$.

In particular, it follows from the theorem that, as $\varepsilon \to 0$, G_ε converges to G_0 uniformly together with $k + l$ derivatives on any closed set not containing the points x_1, x_2, ξ, and together with $\bar{k} \equiv \min(k_1, k_2)$ derivatives on the interval $[x_1, x_2]$.

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Note: Figure translations are in progress. See original paper for figures.

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