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Abstract

Full Text

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SOLUTION OF LINEAR PROGRAMMING PROBLEMS BY THE METHOD OF ELIMINATION OF UNKNOWNS

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1. Let L be an arbitrary real linear space and

$$\begin{aligned} f_j(x) - a_j &\leqslant 0 \qquad (j = 1, 2, \dots, m'), \\ f_j(x) - a_j &< 0 \qquad (j = m' + 1, \dots, m) \end{aligned} \tag{1}$$

be some system of linear inequalities over L, i.e., a system in which $f_j(x)$ $(j = 1, 2, \dots, m)$ are real linear functions defined on L, and a_j $(j = 1, 2, \dots, m)$ are real numbers.

By U we denote some subspace of L, by s the rank of the system of functions $f_j(x)$ $(j = 1, 2, \dots, m)$ restricted to the subspace U, and by

$$t_1 f_{j_1}(x) + \dots + t_{s+1} f_{j_{s+1}}(x) \tag{2}$$

an arbitrary positive (i.e., with $t_1 \geqslant 0, \dots, t_{s+1} \geqslant 0$ and $t_1 + \dots + t_{s+1} > 0$) linear combination, comprising $s + 1$ functions $f_j(x)$, of which s are linearly independent on U. Two combinations (2) that differ only by a positive numerical factor will be regarded as not essentially different.

To an arbitrary combination (2) identically equal to zero on U, we associate the inequality

$$t_1 f_{j_1}(x) + \dots + t_{s+1} f_{j_{s+1}}(x) - (t_1 a_{j_1} + \dots + t_{s+1} a_{j_{s+1}}) < 0$$

in the case when at least one of the functions $f_j(x)$ $(j = m' + 1, \dots, m)$ enters it with a nonzero coefficient, or the inequality

$$t_1 f_{j_1}(x) + \dots + t_{s+1} f_{j_{s+1}}(x) - (t_1 a_{j_1} + \dots + t_{s+1} a_{j_{s+1}}) \leqslant 0$$

otherwise.

The system of such inequalities, corresponding to all possible essentially different combinations (2) identically equal to zero on U, will be called the U-convolution of system (1). For system (1) containing no strict inequalities ($m' = m$), the definition of the U-convolution was given earlier in work [1].

If system (1) is such that from the functions entering it one cannot form even a single positive linear combination (2) identically equal to zero on U, then we shall agree to say that the U-convolution of system (1) is empty.

If V is some direct complement of the subspace U in L, then a nonempty U-convolution of system (1) may be regarded as a system of linear inequalities over the space V ($x \in V$). Its U'-convolution for an arbitrary subspace U' of V will be called the $(U; U')$-convolution of system (1). Similarly one can define the $(U; U'; U'')$-convolution, and so on. All possible U-convolutions of system (1) will be called simple convolutions, and convolutions of its convolutions—repeated convolutions. If the U-convolution of system (1) is empty, then in this case as well one may speak of a repeated convolution, considering it empty.

Some simple or repeated convolution of system (1) will be called complete if its inequalities do not contain nonzero functions, or if it is empty.

In the case of the n-dimensional space R^n, system (1) takes the form

$$\begin{aligned} f_j(x) - a_j &= a_{j1}x_1 + \dots + a_{jn}x_n - a_j \leqslant 0 \qquad (j = 1, 2, \dots, m'), \\ f_j(x) - a_j &= a_{j1}x_1 + \dots + a_{jn}x_n - a_j < 0 \qquad (j = m' + 1, \dots, m). \end{aligned} \tag{3}$$

If U is the subspace generated in R^n by the unit vectors of the coordinate axes $X_{i_1}, \dots, X_{i_k}$, then on it we have

$$f_j(x) = a_{ji_1}x_{i_1} + \dots + a_{ji_k}x_{i_k} \qquad (j = 1, 2, \dots, m).$$

In this case we shall call the U-convolution of system (3) a **convolution with respect to the set of unknowns** $x_{i_1}, \dots, x_{i_k}$, or an $(x_{i_1}, \dots, x_{i_k})$-convolution. The simplest is a convolution with respect to one unknown, an x_i-convolution.* If a_{pi} is an arbitrary positive coefficient and a_{qi} an arbitrary negative coefficient of x_i in system (3), then the x_i-convolution of system (3) will contain (in accordance with the definition of a U-convolution of system (1) given above) one of the two inequalities

$$(a_{pi}f_q(x) - a_{qi}f_p(x)) - (a_{pi}a_q - a_{qi}a_p) \leqslant 0$$

or

$$(a_{pi}f_q(x) - a_{qi}f_p(x)) - (a_{pi}a_q - a_{qi}a_p) < 0.$$

In addition to such inequalities, the x_i-convolution will contain all inequalities of system (3) with coefficients of x_i equal to zero.

Theorem 1. *If system (1) is consistent (i.e., has a solution in L), then every nonempty simple convolution of it is consistent. If at least one simple convolution of system (1) is consistent or empty, then system (1) is consistent.*

Corollary. *System (1) is consistent when at least one of its complete convolutions is consistent or empty. Every complete convolution of a consistent system (1) is consistent or empty.*

Remark. If $v_0 \in V$ $(L = U + V)$ is some solution of the U-convolution of system (1), then, finding for it some solution u_0 of the system

$$\begin{aligned} &f_j(u) - (a_j - f_j(v_0)) \leqslant 0 \quad (j = 1, 2, \dots, m'), \\ &f_j(u) - (a_j - f_j(v_0)) < 0 \quad (j = m' + 1, \dots, m), \end{aligned}$$

we obtain the solution $x_0 = u_0 + v_0$ of system (1).

2. Let $f^{(i)}(x)$ $(i = 1, 2, \dots, s)$ be some system of real linear functions defined on L. The upper bound of the values of the parameter t for which the system

$$\begin{aligned} f_j(x) - a_j &\leqslant 0 \quad (j = 1, 2, \dots, m'), \\ f_j(x) - a_j &< 0 \quad (j = m' + 1, \dots, m), \\ -f^{(i)}(x) + t &\leqslant 0 \quad (j = 1, 2, \dots, s) \end{aligned} \tag{4}$$

is consistent (here system (1) is assumed to be consistent) will be called the **upper-minimal** value of the given system of functions on the set of solutions of system (1).

Theorem 2. *Let system (1) be consistent and let M be the set of its solutions. If the system of real linear functions $f^{(i)}(x)$ $(x \in L)$ $(i = 1, 2, \dots, s)$ has an upper-minimal value on the set M, then every complete convolution of system (4) contains the parameter t (i.e., in it at least one of the coefficients of t is different from zero). If some complete convolution of system (4) contains the parameter t, then the system of functions under consideration has an upper-minimal value on the set M; the latter coincides with the upper bound of the values of the parameter t satisfying such a convolution, and is attained on the set M if and only if among these values there exists a greatest one.*

Let $F_1(x), \dots, F_p(x)$ be some real linear functions on L. The vector $(u_1, \dots, u_p)$ with $u_k = F_k(x)$ $(k = 1, 2, \dots, p)$, for arbitrary x satisfying system (1), will be called the **parameter vector** of system (1). Using the corollary to Theorem 1, it is not difficult to establish that a certain real vector $(u_1^0, \dots, u_p^0)$ is the value of the vector $(u_1, \dots, u_p)$ of the parameters of system (1) if and only if

[continues]

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* Such a convolution was considered in paper (^{2}).

when its coordinates satisfy at least one complete convolution (with respect to x) of the system

$$\begin{aligned} f_j(x) - a_j &\leqslant 0 \quad (j = 1, 2, \dots, m'),\\ f_j(x) - a_j &< 0 \quad (j = m' + 1, \dots, m),\\ F_k(x) - u_k &\leqslant 0 \quad (k = 1, 2, \dots, p),\\ -F_k(x) + u_k &\leqslant 0 \quad (k = 1, 2, \dots, p). \end{aligned}$$

To an arbitrary value $(u_1^0, \dots, u_p^0)$ of the vector of parameters $(u_1, \dots, u_p)$ of system (1) there corresponds uniquely the upper bound $T(u_1^0, \dots, u_p^0)$ of the values of the parameter t ($+\infty$ is not excluded here) for which system (4) has a solution x satisfying the conditions $F_1(x) = u_1^0, \dots, F_p(x) = u_p^0$. By the corollary of Theorem 1, this bound coincides with the upper bound of the values of the parameter t for which at least one complete convolution $S(u_1, \dots, u_p, t)$ of the system is consistent,

$$\begin{aligned} f_j(x) - a_j &\leqslant 0 \quad (j = 1, 2, \dots, m'),\\ f_j(x) - a_j &< 0 \quad (j = m' + 1, \dots, m),\\ F_k(x) - u_k &\leqslant 0 \quad (k = 1, 2, \dots, p),\\ -F_k(x) + u_k &\leqslant 0 \quad (k = 1, 2, \dots, p),\\ -f^{(i)}(x) + t &\leqslant 0 \quad (i = 1, 2, \dots, s), \end{aligned}$$

taken for $u_k = u_k^0$ $(k = 1, 2, \dots, p)$.

Thus, the dependence of the upper-minimal value of interest to us on the parameters $u_1, \dots, u_p$ is completely determined by the convolution $S(u_1, \dots, u_p, t)$. The more general question of the dependence of this value on parameters that may enter into the free terms of system (1) is solved analogously.

3. By an **elementary transformation** of system (3) we shall mean bringing it to the form

$$a_{j1}x_1 + \dots + (a_{jk} + \alpha a_{jl})x_k + \dots + a_{jl}u_l + \dots + a_{jn}x_n - a_j \leqslant 0$$
$$(j = 1, 2, \dots, m'),$$

$$a_{j1}x_1 + \dots + (a_{jk} + \alpha a_{jl})x_k + \dots + a_{jl}u_l + \dots + a_{jn}x_n - a_j < 0$$
$$(j = m' + 1, \dots, m),$$

where α is an arbitrary real number and $u_l = x_l - \alpha x_k$. If some elementary transformation of system (3) does not affect its terms with some x_p, then the upper and lower bounds of the p-th coordinate of its solutions ($+\infty$ and $-\infty$ are not excluded here) do not change in passing to the transformed system. Therefore, when finding the upper-minimal value of the system of functions

$f^{(i)}(x) = b_{i1}x_1 + \ldots + b_{in}x_n$ $(i = 1, 2, \ldots, s)$ on the set of solutions of system (3), repeated convolution of the corresponding system (4) with respect to the unknowns x_i may be alternated with suitably chosen elementary transformations of the resulting convolutions. Such alternation in many cases makes it possible to substantially reduce the number of inequalities in the final convolution, which contains only the parameter t.

Example. Find the upper-minimal value T of the pair of functions $11x_3 + 2x_4,\ 2x_3 + x_4$ on the set of solutions of the system

$$\begin{aligned} x_1 + x_2 - x_3 - x_4 - 1 &< 0, \\ -2x_1 + x_2 + x_3 - x_4 + 0.5 &< 0, \\ 6x_1 - x_2 - 2x_3 + x_4 - 4 &\leqslant 0, \\ -x_1 - x_2 + x_3 - x_4 - 0 &\leqslant 0, \\ 4x_1 - x_2 + x_3 + x_4 - 5 &\leqslant 0, \\ -x_1 + 2x_2 - x_3 + x_4 - 1 &\leqslant 0. \end{aligned}$$

In accordance with Theorem 2, this question is solved by convolving the system obtained by supplementing this system with the two inequalities

$$\begin{aligned} -11x_3 - 2x_4 + t &\leqslant 0, \\ -2x_3 - x_4 + t &\leqslant 0. \end{aligned}$$

Subjecting the augmented system to the elementary transformation determined by the numbers $k = 1,\ l = 2,\ a = 3$, we obtain:

$$\begin{aligned} 4x_1 + u_2 - x_3 - x_4 - 1 &< 0, \\ x_1 + u_2 + x_3 - x_4 + 0.5 &< 0, \\ 3x_1 - u_2 - 2x_3 + x_4 - 4 &\leqslant 0, \\ -4x_1 - u_2 + x_3 - x_4 - 0 &\leqslant 0, \\ x_1 - u_2 + x_3 + x_4 - 5 &\leqslant 0, \\ 5x_1 + 2u_2 - x_3 + x_4 - 1 &\leqslant 0, \\ -11x_3 - 2x_4 + t &\leqslant 0, \\ -2x_3 - x_4 + t &\leqslant 0, \end{aligned}$$

where $u_2 = x_2 - 3x_1$.

Eliminating this system with respect to x_1, we obtain the system

$$
\begin{aligned}
-2x_4 - 1 &< 0,\\
3u_2 + 5x_3 - 5x_4 + 2 &< 0,\\
-7u_2 - 5x_3 + x_4 - 16 &\leqslant 0,\\
-5u_2 + 5x_3 + 3x_4 - 20 &\leqslant 0,\\
3u_2 + x_3 - x_4 - 4 &\leqslant 0,\\
-11x_3 - 2x_4 + t &\leqslant 0,\\
-2x_3 - x_4 + t &\leqslant 0.
\end{aligned}
$$

Subjecting the latter to the obvious elementary transformation, we bring it to the form:

$$
\begin{aligned}
-2x_4 - 1 &< 0,\\
3v_2 + 5x_3 - 2x_4 + 2 &< 0,\\
-7v_2 - 5x_3 - 6x_4 - 16 &\leqslant 0,\\
-5v_2 + 5x_3 - 2x_4 - 20 &\leqslant 0,\\
3v_2 + x_3 + 2x_4 - 4 &\leqslant 0,\\
-11x_3 + 2x_4 + t &\leqslant 0,\\
-2x_3 - x_4 + t &\leqslant 0,
\end{aligned}
$$

where $v_2 = u_2 - x_4$.

Eliminating this system with respect to x_4, then with respect to v_2, and, finally, with respect to x_3, we obtain successively the systems:

$$
\begin{array}{rrr}
\begin{aligned}
3v_2 + x_3 - 5 &< 0,\\
6v_2 + 6x_3 - 2 &< 0,\\
2v_2 - 2x_3 - 28 &\leqslant 0,\\
-2v_2 + 6x_3 - 24 &\leqslant 0,\\
3v_2 - 10x_3 - 4 + t &\leqslant 0,\\
3v_2 - 3x_3 - 4 + 2t &\leqslant 0;
\end{aligned}
&
\begin{aligned}
20x_3 - 82 &< 0,\\
24x_3 - 74 &< 0,\\
4x_3 - 52 &\leqslant 0,\\
-2x_3 - 80 + 2t &\leqslant 0,\\
12x_3 - 80 + 4t &\leqslant 0;
\end{aligned}
&
\begin{aligned}
-882 + 20t &< 0,\\
-1034 + 24t &< 0,\\
-212 + 4t &\leqslant 0,\\
-560 + 16t &\leqslant 0.
\end{aligned}
\end{array}
$$

The last system gives $T = 35$. Since $t = 35$ satisfies it, by Theorem 2 the upper-minimal value found, $T = 35$, is attained on the set of solutions of the original system.

Substituting $t = 35$ into the system containing x_3 and t, we obtain $x_3 = -5$. Substituting $x_3 = -5$ and $t = 35$ into the system containing v_2, x_3, and t, we obtain $v_2 = -27$. Substituting $v_2 = -27$, $x_3 = -5$, and $t = 35$ into the preceding system, we find $x_4 = 45$. From the relation $v_2 = u_2 - x_4$ we have $u_2 = 18$. Substituting $u_2 = 18$, $x_3 = -5$, $x_4 = 45$, and $t = 35$ into the system with which the elimination began, we find $x_1 = -17$. From the relation

$u_2 = x_2 - 3x_1$ we have $x_2 = -33$. Thus the upper-minimal value $T = 35$ is attained for the solutions $(-17, -33, -5, 45)$ of the original system.

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CITED LITERATURE

[1] S. N. Chernikov, DAN, **134**, No. 3, 518 (1960). [2] H. W. Kuhn, *Am. Math. Monthly*, **63**, 4, 27 (1956).

Note: Figure translations are in progress. See original paper for figures.