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Abstract

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On an Internal Characterization of Topologically Complete Spaces in the Sense of E. Čech

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A space* P is called **topologically complete in the sense of E. Čech** (hereafter simply **complete**) if it is a G_δ-set in its Čech compactification $\beta(P)$. In the paper (2) the author succeeded, by means of so-called complete sequences of open covers, in giving an internal characterization of complete spaces. In the present note an internal characterization of complete spaces is given by means of a "diameter," and also by means of a certain ordering of a system of open subsets.

Definition 1. By a **diameter** on a space P we shall mean any nonnegative function** d, defined on the system $\exp P$ of all subsets of the set P and satisfying the following conditions:

(d1) If $M \subset N \subset P$, then $d(M) \leq d(N)$.

(d2) For every $M \subset P$, $d(M)$ is the greatest lower bound of the set of all $d(U)$, where U is an open set containing M.

(d3) $d((x)) = 0$ for every point $x \in P$.

Example 1. If φ is a pseudometric on the space P and if $d(M)$ denotes the diameter of the set M in the usual sense, i.e. $d(M) = \sup \varphi(x, y)$, $x \in M$, $y \in M$, then d is a diameter in the sense of our definition. We shall call it the **diameter generated by the pseudometric** φ. If f is a continuous function on the space P and if $d(M)$ denotes the oscillation of the function f on the set M, then d is a diameter generated by the function f.

Definition 2. Let d be a diameter on the space P. A centered system $\mathfrak{A}$ of subsets of the space P will be called d**-shrinking** (or a d**-system**) if $\inf\{d(A);\ A \in \mathfrak{A}\} = 0$. The diameter d will be called **complete** if the intersection of the closures of the sets of every d-system is nonempty.

The fundamental property of a complete diameter is its nonextendability. We shall call a space R an **extension** of the space P if $R \supset P$ and $\overline{P} = R$; if, in addition, $R = P$, then R is called an **identical** extension of the space P.

Definition 3. A diameter d on the space P will be called **nonextendable** if every extension R of the space P on which there exists a diameter D such that

$d(M) = D(M)$ for $M \subset P$ is identical.

Theorem 1. *In order that a diameter on a space be nonextendable, it is necessary and sufficient that it be complete.*

Proof. Suppose that D is a diameter on a space R, $P \subset R$, $\overline{P} = R$, $P \neq R$, and d is the same function D, considered on $\exp P$. Let $x \in R - P$. Then the system $\mathfrak{B}$ of all $A \cap P$, where

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* "Space" throughout the paper means a completely regular topological space.

** By a "function" we shall always understand a real-valued function (which, generally speaking, also assumes the values $-\infty$ and ∞).

A—a neighborhood of x in R, is a d-system, and the intersection of the closures in P of the sets from $\mathfrak{B}$ is empty. Thus, the diameter d is not complete.

Now let d be an incomplete diameter on the space P. There exists a d-system $\mathfrak{A}$ such that the intersection of the closures of the sets from $\mathfrak{A}$ is empty. Take some point x lying in the intersection of the closures of these sets in the Čech extension $\beta(P)$, and consider the space $R = P \cap (x) \subset \beta(P)$. For every open $U \subset R$, put $D(U) = d(U \cap P)$, and for every $M \subset R$ put

$$D(M) = \inf\{D(U);\ U \supset M,\ U \text{ open}\}.$$

It turns out that D is a diameter on the set R. Clearly, $d(M) = D(M)$ for $M \subset P$.

Theorem 2. *If there exists a complete diameter on the space P, then P is a G_δ-set in every extension of itself.*

Proof. Let d be a complete diameter on the space P, and let R be an extension of the space P. For every natural n, let U_n be the union of all open $U \subset R$ satisfying the inequality $d(P \cap U) < \frac{1}{n}$. Clearly, there exists an extension of the diameter d to $\bigcap_{n=1}^{\infty} U_n$. Thus,

$$\bigcap_{n=1}^{\infty} U_n = P.$$

Theorem 3. *If there exists a complete diameter on the space P, then there exists a complete diameter on every G_δ-subset of the space P.*

Proof. Let d be a complete diameter on the space P, and let

$$R = \bigcap_{n=1}^{\infty} U_n,$$

where U_n is an open subset of the space P. For every open $U \subset R$, put

$$d_1(U) = \inf\left\{\frac{1}{n};\ \overline{U} \cap R \subset U_n\right\},$$

$$D(U) = \max[d(U), d_1(U)].$$

For every $M \subset R$, put

$$D(M) = \inf\{D(U),\ U \text{ open in } R,\ U \supset M\}.$$

It turns out that D is a complete diameter on the space R.

Corollary. *The space P is complete if and only if there exists a complete diameter on P.*

It is not difficult to prove the following two theorems.

Theorem 4. *A diameter d on the space P is complete if and only if the following two conditions are satisfied:*

(1) *If $d(M) = 0$, then the closure of the set M is compact.*

(2) *The intersection of the closures of the sets of any countable d-system is nonempty.*

Theorem 5. *Let d be a complete diameter on the space P. For every $M \subset P$, let $d_1(M)$ be the greatest lower bound of the set of all $\varepsilon > 0$ for which there exists a finite number of sets $M_1, \dots, M_k$ such that*

$$M \subset \bigcup_{i=1}^{k} M_i \quad \text{and} \quad d(M_i) < \varepsilon.$$

Then d_1 is a complete diameter on the space P.

A sequence $\{\mathfrak{A}_n\}$ of open covers of the space P is called **complete** if, for every centered system of sets $\mathfrak{A}$ such that $\mathfrak{A} \cap \mathfrak{A}_n \neq \emptyset$ for all $n = 1, 2, \dots$, the intersection of the closures of the sets from $\mathfrak{A}$ is nonempty.

Let $\alpha = \{\mathfrak{A}_n\}$ be a sequence of open covers of the space P satisfying the following two conditions:

(p1) $\mathfrak{A}_n \supset \mathfrak{A}_{n+1},\ n = 1, 2, \dots$.

(p2) If A is open and $A \subset B \in \mathfrak{A}_n$, then $A \in \mathfrak{A}_n$.

For every open $U \subset P$ put $d(U) = 1$, if $U \notin \mathfrak{A}_1$. If, however, $U \in \mathfrak{A}_1$, put

$$d(U) = \inf\left\{\frac{1}{n};\ U \in \mathfrak{A}_n\right\}.$$

Finally, for any $M \subset P$ put

$$d(M) = \inf\{d(U);\ U \supset M,\ U \text{ open}\}.$$

Obviously, d is a diameter on the space P. It turns out that α is a complete sequence if and only if d is a complete diameter.

Now let d be a diameter on the space P. Let $\mathfrak{A}_n$ be the system of all open sets U for which $d(U) < \frac{1}{n}$. It turns out that the sequence $\{\mathfrak{A}_n\}$ of open coverings is complete if and only if d is a complete diameter.

If a complete sequence of open coverings is given, it is not difficult to construct a complete sequence satisfying conditions (p1) and (p2). Thus, a complete sequence of open coverings exists if and only if a complete diameter exists.

In proofs it is convenient to use additive complete sequences satisfying conditions (p1) and (p2), since if $\{\mathfrak{A}_n\}$ is a complete sequence and if $\mathfrak{B}_n$ consists of the unions of all finite subsystems of the covering $\mathfrak{A}_n$, then $\{\mathfrak{B}_n\}$ is also a complete sequence.

Let d be a complete diameter on the space P. Define an ordering of open sets $>$ so that $A > B$ if and only if $A \supset B$ and

$$2d(B) \leqslant \min[d(A), 1].$$

It turns out that the following four conditions are fulfilled:

(u1) If $A > B$, then $A \supset B$.

(u2) If C and D are open, $C \supset A$, $D \subset B$, and $A > B$, then $C > D$.

(u3) If A is an open set, then the system of all $B < A$ is a basis for the open subsets of the set A.

(u4) If $\mathfrak{A}$ is a centered system of subsets of the space P and if for every natural number n there exist $A_1, \dots, A_{n+1}$ such that $A_i > A_{i+1}$, $i = 1, \dots, n$, and A_{n+1} contains some $A \in \mathfrak{A}$, then the intersection of the closures of the sets from $\mathfrak{A}$ is nonempty.

Now let $>$ be some ordering of open sets satisfying conditions (u1)–(u4). For every $M \subset P$ let $d(M)$ be the lower bound of all $\frac{1}{n}$ for which there exist $A_1, \dots, A_{n+1}$ such that $A_{n+1} \supset M$ and $A_i > A_{i+1}$, $i = 1, \dots, n$. If no such n exists, put $d(M) = 1$. It turns out that d is a complete diameter on the space P.

Thus the following is true:

Theorem 6. The following properties of the space P are equivalent:

(1) P is a complete space.

(2) P is a G_δ-set in some complete space.

(3) P is a G_δ-set in every one of its extensions.

(4) There exists a complete diameter on the space P.

(5)* There exists a complete sequence of open coverings of the space P.

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* The equivalence of completeness to property (5) was proved in the work of A. Arhangel' skii ([3]), submitted for publication on 18 V 1960. *P. S. Aleksandrov.*

(6) There exists an ordering of the open sets of the space satisfying conditions (u1)–(u4).

Theorem 7. The following properties of a space are equivalent:

(1) There exists a perfect* mapping of the space P onto some complete metric space.

(2) There exists on P a complete diameter generated by a pseudometric.

(3) P is a complete space and every additive open covering of the space P is normal.**

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References

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[2] Z. Frolík, Czechoslovak Math. J., 85, 359 (1960).
[3] A. Arhangel' skii, Vestn. Mosk. Univ., Ser. Math. and Mech., No. 1 (1961).
[4] Z. Frolík, Czechoslovak Math. J., 84, 63 (1959).

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* A continuous and closed mapping is called **perfect** if the full preimages of all points are bicompact.

** A covering $\mathfrak{A}$ is called **normal** if there exists a sequence $\{\mathfrak{A}_n\}$ of open coverings such that $\mathfrak{A}_1$ is inscribed in $\mathfrak{A}$ and $\mathfrak{A}_{n+1}$ is star-inscribed in $\mathfrak{A}_n$, $n = 1, 2, \ldots$.

Note: Figure translations are in progress. See original paper for figures.