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Abstract

Full Text

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ON BOUNDARY PROPERTIES OF FUNCTIONS FROM "WEIGHTED" CLASSES $W^1_{\vec{\alpha},p}$

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In the present paper we determine necessary and sufficient conditions satisfied by a function on the boundary of a domain if each of its first derivatives is summable in the domain with a certain weight degenerating on the boundary, in general different for each derivative.

Definition. Let R^{n-1} be the $(n-1)$-dimensional space of points $(x_1, \dots, x_{n-1})$, and let R^n_0 be the half-space $(x_n \geq 0,\ x(x_1, \dots, x_{n-1}) \in R^{n-1})$. We shall say that a function f belongs to the class $W^1_{\vec{\alpha},p}$

$$(1 < p < \infty, \quad \vec{\alpha} = (\alpha_1, \dots, \alpha_n), \quad -1 < \alpha_i < p-1, \quad i = 1, \dots, n),$$

if the function f has all first generalized Sobolev derivatives ([1]) and

$$\|f\|^p_{W^1_{\vec{\alpha},p}} = \int_{R^n_0} |f|^p \, dR^n_0 + \int_{R^n_0} \sum_{i=1}^{n} x_n^{\alpha_i} \left| \frac{\partial f}{\partial x_i} \right|^p dR^n_0 < \infty. \tag{1}$$

Following L. N. Slobodetskii ([2]), we shall also introduce the classes $W^r_{x,p}$ $(1 < p < \infty,\ x = (x_1, \dots, x_{n-1}),\ r = (r_1, \dots, r_{n-1}),\ 0 < r_i < 1)$. We shall consider that $f \in W^r_{x,p}$ if $f \in L_p(R^{n-1})$ and

$$\|f\|^p_{W^r_{x,p}} = \int_{R^{n-1}} |f|^p \, dR^{n-1} + \sum_{i=1}^{n-1} \int_0^\infty \int_{R^{n-1}} \frac{|\Delta_{i,h} f|^p}{h^{1+pr_i}} \, dR^{n-1} \, dh < \infty, \tag{2}$$

where

$$\Delta_{i,h} f = f(x_1, \dots, x_i, \dots, x_{n-1}) - f(x_1, \dots, x_i + h, \dots, x_{n-1}).$$

We shall also say that the function f assumes the value φ on R^{n-1} $(F|_{R^{n-1}} = \varphi)$ if, after a possible modification of F on a set of n-dimensional measure zero,

$$\lim_{x_n \to 0} \int_{R^{n-1}} |F - \varphi|^p \, dR^{n-1} = 0. \tag{3}$$

The following two theorems generalize the corresponding results of A. A. Vasharin ([3]) and P. I. Lizorkin ([4]).

Theorem 1 (direct). Let $f \in W^1_{\vec{\alpha},p}$. Then

$$f\big|_{R^{n-1}} = \varphi \in W^{\vec{\beta}}_{\vec{x},p}, \qquad \beta_i = \frac{p-1-\alpha_n}{p-\alpha_n+\alpha_i} \quad (i = 1, \dots, n-1),$$

$$\|\varphi\|_{W^{\vec{\beta}}_{\vec{x},p}} \le c\|f\|_{W^1_{\vec{\alpha},p}}, \tag{4}$$

where the constant c does not depend on f.

Theorem 2 (converse). Let $\varphi \in \vec{W}^{\beta}_{x,p}$; let α_n be arbitrary in the interval $(-1, p-1)$, and let

$$\alpha_i = \alpha_n - p + \frac{p-\alpha_n-1}{\beta_i} \quad (i = 1, \dots, n-1).$$

Then there exists an infinitely differentiable function f, defined on R^n_0, such that

$$f\big|_{R^{n-1}} = \varphi, \qquad \|f\|_{W^1_{\alpha,p}} \le c\|\varphi\|_{\vec{W}^{\beta}_{x,p}}, \tag{5}$$

where the constant c does not depend on φ.

We give brief proofs of the theorems.

Proof of Theorem 1. The existence of boundary values φ and their membership in $L_p(R^{n-1})$ follow from the corresponding results of L. D. Kudryavtsev ([5]).

We prove (4). Let

$$\gamma_i = \frac{p}{p-\alpha_n+\alpha_i}, \qquad \beta_i = \frac{p-1-\alpha_n}{p-\alpha_n+\alpha_i}.$$

We have

$$\|\varphi\|^p_{\vec{W}^{\beta}_{x,p}} = \sum_{i=1}^{n-1} \int_0^\infty \int_{R^{n-1}} \frac{|\Delta_{i,h}\varphi|^p}{h^{1+p\beta_i}}\, dR^{n-1}\, dh \le$$

$$\le c \sum_{i=1}^{n-1} \int_0^\infty \int_{R^{n-1}} \frac{|f(x_1, \dots, x_i, \dots, x_{n-1}, h^{\gamma_i}) - f(x_1, \dots, x_i, \dots, x_{n-1}, 0)|^p}{h^{1+p\beta_i}}\, dR^{n-1}\, dh +$$

$$+ c \sum_{i=1}^{n-1} \int_0^\infty \int_{R^{n-1}} \frac{|f(x_1, \dots, x_i + h, \dots, x_{n-1}, h^{\gamma_i}) - f(x_1, \dots, x_i, \dots, x_{n-1}, h^{\gamma_i})|^p}{h^{1+p\beta_i}}\, dR^{n-1}\, dh +$$

$$+ c \sum_{i=1}^{n-1} \int_0^\infty \int_{R^{n-1}} \frac{|f(x_1, \dots, x_i + h, \dots, x_{n-1}, h^{\gamma_i}) - f(x_1, \dots, x_i + h, \dots, x_{n-1}, 0)|^p}{h^{1+p\beta_i}}\, dR^{n-1}\, dh \le$$

$$\leq c\sum_{i=1}^{n-1}\int_0^\infty\int_{R^{n-1}}\left[\left|\int_0^{h^{\gamma_i}}\frac{\partial f(x,u)}{\partial x_n}\,du\right|^p+h^p\left|\frac{\partial f(x,h^{\gamma_i})}{\partial x_i}\right|^p\right]h^{-1-p\beta_i}\,dR^{n-1}\,dh=$$

(substitution $v=h^{\gamma_i}$)

$$=c_1\sum_{i=1}^{n-1}\int_0^\infty\int_{R^{n-1}}\left[\left(\int_0^{v}\frac{\partial f(x,u)}{\partial x_n}\,du\right)^p+v^{p/\gamma_i}\left|\frac{\partial f(x,v)}{\partial x_i}\right|^p\right]v^{-(1+p\beta_i/\gamma_i)}\,dR^{n-1}\,dh\leq$$

(Hardy' s inequality ([6]))

$$\leq c_2\sum_{i=1}^{n}\int_0^\infty\int_{R^{n-1}}\left(x_n^{p-1-p\beta_i/\gamma_i}\left|\frac{\partial f}{\partial x_n}\right|^p+x_n^{(p-p\beta_i)/\gamma_i-1}\left|\frac{\partial f}{\partial x_i}\right|^p\right)dR^{n-1}\,dx_n=$$

$$=c_2\sum_{i=1}^{n}\int_0^\infty\int_{R^{n-1}}\left(x_n^{\alpha_n}\left|\frac{\partial f}{\partial x_n}\right|^p+x_n^{\alpha_i}\left|\frac{\partial f}{\partial x_i}\right|^p\right)dR^{n-1}\,dx_n.$$

Proof of Theorem 2. Let α_n be arbitrary fixed from the interval $(-1,p-1)$. Put

$$\gamma_i=\frac{p-\alpha_n-1}{p\beta_i}>0.$$

Construct the function

$$f=k\int_{R^{n-1}}\varphi(t)\prod_{i=1}^{n-1}\frac{x_n^{\gamma_i}}{(t_i-x_i)^2+x_n^{2\gamma_i}}\,dR^{n-1},$$

where

$$k=\left[\int_{R^{n-1}}\prod_{i=1}^{n-1}\frac{1}{1+u_i^2}\,dR^{n-1}\right]^{-1}.$$

We shall show that the function f satisfies the conditions of the theorem. It is easy to see that $f|_{R^{n-1}}=\varphi$. We also have, denoting by Π' the product Π with one factor omitted,

$$\frac{\partial f}{\partial x_n}=k\int_{R^{n-1}}\varphi(t)\sum_{j=1}^{n-1}{\prod_{i=1}^{n-1}}'\frac{x_n^{\gamma_i}}{(t_i-x_i)^2+x_n^{2\gamma_i}}\,\frac{\partial}{\partial x_n}\left[\frac{x_n^{\gamma_j}}{(t_j-x_j)^2+x_n^{2\gamma_j}}\right]dR^{n-1}=$$

$$=k\int_{R^{n-1}}\varphi(t)\sum_{j=1}^{n-1}\prod_{i=1}^{n-1}\frac{x_n^{\gamma_i}}{(t_i-x_i)^2+x_n^{2\gamma_i}}\,\gamma_j\frac{\partial^2}{\partial u^2}\left[\ln\left[\frac{1}{(t_j-x_j)^2+u^2}\right]^{1/2}\right]_{u=x_n^{\gamma_i}}x_n^{\gamma_j-1}\,dR^{n-1}=$$

$$= -k \int_{R^{n-1}} \sum_{j=1}^{n-1} \gamma_j \left[\varphi(t_1, \dots, t_j, \dots, t_{n-1}) - \varphi(t_1, \dots, x_j, \dots, t_{n-1})\right] \times$$

$$\times \prod_{i=1}^{n-1}{}' \frac{x_n^{\gamma_i}}{(t_i - x_i)^2 + x_n^{2\gamma_i}} \frac{\partial^2}{\partial t_j^2} \left[\ln \left[\frac{1}{(t_j - x_j)^2 + x_n^{2\gamma_i}}\right]^{1/2}\right] x_n^{\gamma_j - 1}\, dR^{n-1};$$

$$\left|\frac{\partial f}{\partial x_n}\right| \le c \sum_{j=1}^{n-1} \int_{R^{n-1}} |\Delta_{j,t_j}\varphi| \prod_{i=1}^{n-1}{}' \frac{x_n^{\gamma_i}}{(t_i - x_i)^2 + x_n^{2\gamma_i}} \frac{x_n^{\gamma_j - 1}}{t_j^2 + x_n^{2\gamma_j}}\, dR^{n-1};$$

$$\int_{R_0^n} x_n^{\alpha_n} \left|\frac{\partial f}{\partial x_n}\right|^p dR_0^n \le$$

$$\le c \int_{R_0^n} x_n^{\alpha_n} \sum_{j=1}^{n-1} \left[\int_{R^{n-1}} |\Delta_{j,t_j}\varphi| \prod_{i=1}^{n-1}{}' \frac{x_n^{\gamma_i}}{(t_i - x_i)^2 + x_n^{2\gamma_i}} \frac{x_n^{\gamma_j - 1}}{t_j^2 + x_n^{2\gamma_j}}\, dR^{n-1}\right]^p dR_0^n \le$$

$$\le c \sum_{j=1}^{n-1} \int_{R_0^n} x_n^{\alpha_n} \int_{R^{n-1}} |\Delta_{j,t_j}\varphi|^p \prod_{i=1}^{n-1}{}' \frac{x_n^{p\gamma_i}}{(t_i - x_i)^2 + x_n^{2\gamma_i}} \frac{x_n^{p(\gamma_j - 1)}}{[t_j^2 + x_n^{2\gamma_j}]^{\varepsilon p/2}}\, dR^{n-1} \times$$

$$\times \left[\int_{R^{n-1}} \prod_{i=1}^{n-1}{}' \frac{1}{(t_i - x_i)^2 + x_n^{2\gamma_i}} \frac{1}{[t_j^2 + x_n^{2\gamma_j}]^{(2-\varepsilon)q/2}}\, dR^{n-1}\right]^{p/q} dR_0^n \le$$

$$\left(\frac{1}{p} + \beta_j < \varepsilon < 1 + \frac{1}{p}\right)$$

$$\le c_1 \sum_{j=1}^{n-1} \int_{-\infty}^{\infty} \int_{R^{n-1}} |\Delta_{j,t_j}\varphi|^p \int_0^{\infty} \frac{x_n^{\alpha_n - \gamma_j - p + p\gamma_j\varepsilon}}{(t_j^2 + x_n^{2\gamma_j})^{\varepsilon p/2}}\, dx_n\, dR^{n-1}\, dt_j \le$$

$$\left(\text{substitution } \frac{x_n^{\gamma_j}}{t_j} = v\right)$$

$$\le c_2 \sum_{j=1}^{n-1} \int_{-\infty}^{\infty} \int_{R^{n-1}} \frac{|\Delta_{j,t_j}\varphi|^p}{|t_j|^{1+p\beta_j}} \int_0^{\infty} \frac{v^{\theta}}{[1 + v^2]^{\varepsilon p/2}}\, dv\, dR^{n-1}\, dt_j \le$$

$$\left(p\varepsilon > 1 + p\beta_j,\ \theta = \frac{\alpha_n}{\gamma_j} - 1 - \frac{p}{\gamma_j} + p\varepsilon + \frac{1}{\gamma_j} - 1 > -1\right)$$

$$\leq c_2 \sum_{j=1}^{n-1} \int_0^\infty \int_{R^{n-1}} \frac{|\Delta_j h^\varphi|^p}{h^{1+p\beta_j}} \, dR^{n-1} \, dh.$$

Estimates for the derivatives $\partial f / \partial x_i$ $(i = 1, \dots, n-1)$ are obtained similarly.

Theorems 1 and 2 also carry over to bounded domains with sufficiently smooth boundaries.

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Note: Figure translations are in progress. See original paper for figures.