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Abstract

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MATHEMATICS

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THE STRUCTURE OF POLYNOMIAL IDEALS AND THEIR QUOTIENT SPACES IN SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS

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In the study of solutions of partial differential equations with constant coefficients in the whole space, the following problems occupy a central place:

1. For which spaces Φ of ordinary or generalized functions does the equation

$$p(D)u = w, \quad D = \left(i \frac{\partial}{\partial x_1}, \dots, i \frac{\partial}{\partial x_n} \right) \quad (1)$$

have a solution $u \in \Phi$ for every $w \in \Phi$?

2. What is the general form of the solutions of (1) with $w = 0$ in the space Φ ?

Here $p = p(s)$ is a polynomial or polynomial matrix in the variables $s = (s_1, \dots, s_n)$.

The solution of these problems depends essentially on whether the space Φ contains functions growing like e^{cx} , or not. We shall restrict ourselves to the case when Φ is a space of slowly increasing functions or functionals, i.e. increasing at infinity no faster than certain powers of $|x|$. Such spaces, in particular, are the spaces $S^{\beta'}$ and \mathcal{E}^{β} (see ^(1,2)).

L. Hörmander in ⁽³⁾ and S. Łojasiewicz in ⁽⁴⁾ solved problem 1 for some spaces Φ of this kind. Little is known about the solution of problem 2 (see ^(5,6)).

Apply the Fourier transform to the space Φ and to equation (1). The space dual to Φ will be denoted by $\tilde{\Phi}$. Equation (1) passes into the equation

$$p(s)\tilde{u} = \tilde{w}, \quad \tilde{w} \in \tilde{\Phi}. \quad (2)$$

The solvability of this equation in $\tilde{\Phi}$, as is well known, is equivalent to the closedness of the ideal $p\tilde{\Phi}'$ in the space $\tilde{\Phi}'$. If $w = 0$, then also $\tilde{w} = 0$; in this

case every solution of (2) is a continuous functional on $\tilde{\Phi}'/p\tilde{\Phi}'$, and conversely. Thus problem 2 reduces to describing the space conjugate to $\tilde{\Phi}'/p\tilde{\Phi}'$.

We now note that the space $\tilde{\Phi}'$ is a certain space of infinitely differentiable functions on R^n satisfying specified restrictions on growth at infinity (together with all their derivatives), but without restrictions on the growth of derivatives depending on their order. This property is easily verified for the particular cases $\Phi = S^{\beta'}, \mathcal{E}^{\beta}$. Indeed (see ^(1,2)):

$$\tilde{S}^{\beta} = S_{\beta}, \quad \tilde{\mathcal{E}}^{\beta} = \mathcal{E}_{\beta}.$$

It is natural to formulate a somewhat more general problem.

Let \mathcal{E} be some complete space of infinitely differentiable functions of the type described above. It is required to describe the polynomial

the ideal $p\mathcal{E}$ and the quotient space $\mathcal{E}/p\mathcal{E}$ with respect to this ideal. By the ideal $p\mathcal{E}$ we mean the ideal in the space \mathcal{E} generated by the polynomials $p = (p_1, \dots, p_{\sigma})$. (The case $\sigma = 1$ corresponds to one equation (1), the case $\sigma > 1$ to a system of equations.)

To describe the solutions of these problems, we shall carry out some preliminary constructions.

By the local ideal I_s at a point s , generated by the polynomials p in the space \mathfrak{M} of formal power series, we shall mean the space of series of the form

$$p_1(s + \xi)\psi_1[\xi] + \dots + p_{\sigma}(s + \xi)\psi_{\sigma}[\xi],$$

where $\psi_1, \dots, \psi_{\sigma}$ belong to \mathfrak{M} , and the polynomials $p_{\tau}(s + \xi)$, $\tau = 1, \dots, \sigma$, are considered by us as power series $\sum \frac{1}{j!} D^j p_{\tau}(s) \xi^j$.

Let p be some polynomial, and let N_{ν} be the set of its real roots of multiplicity not less than ν , i.e. the set of real solutions of the system

$$D^j p(s) = 0, \quad |j| \leq \nu - 1.$$

We obtain the following chain of varieties nested one in another:

$$R^n = N_0 \supset N_1 \supset \dots \supset N_m \supset N_{m+1} = \emptyset,$$

where m is the order of p . We construct such chains for each polynomial in the set $p = (p_1, \dots, p_{\sigma})$ and for each polynomial that is the least common multiple of some group of polynomials of this set. From each chain choose one variety and form their intersection, which we denote by N_{ν} ; ν is a vector-index whose components are the indices of the corresponding intersected varieties. The indices ν are naturally ordered: $\nu \leq \nu'$, if all components of ν do not exceed the

corresponding components of ν' . In this case $N_\nu \supset N_{\nu'}$. By $N_{\nu+1}$ we denote the union of all varieties $N_{\nu'}$ for which $\nu \leq \nu'$, $\nu \neq \nu'$. Each point $s \in R^n$ belongs to some set of the form $N_\nu \setminus N_{\nu+1}$. Let N be the largest index, i.e. for all ν , $\nu \leq N$.

By N we shall denote the set of common zeros of the polynomials p_1, \dots, p_σ .

Lemma 1. At each point $s \in R^n$ there exists a countable system of operators $\mathcal{D}_\alpha(s, D)$ with polynomial coefficients on each set $N_\nu \setminus N_{\nu+1}$, and a countable system of operators $G_\tau^i(s, D)$, $1 \leq \tau \leq \sigma$, whose coefficients on each set $N_\nu \setminus N_{\nu+1}$ are rational functions whose denominators do not vanish, possessing the following properties:

1. If ψ belongs to \mathfrak{M} , then the fulfillment of all equalities $\mathcal{D}_\alpha(s, D)\psi = 0$ is equivalent to ψ belonging to I_s .
2. If ψ belongs to the ideal I_s , then

$$\psi[\xi] = \sum_{\tau} p_{\tau}(s + \xi) \sum_i \xi^i G_{\tau}^i(s, D)\psi.$$

The operators \mathcal{D}_α and G_τ^i are not determined uniquely by this lemma; however, we can choose them so that the following two lemmas hold.

Lemma 2. There exist such functions $\chi_{i\alpha}(s)$, rational on each set $N_\nu \setminus N_{\nu+1}$ with denominators not vanishing there, that for any infinitely differentiable function ψ , points $s \in N_\nu \setminus N_{\nu+1}$, $s + \xi \in R^n$, integer $k \geq 0$, and index α , the inequality holds

$$\left| \mathcal{D}_\alpha(s + \xi, D_s)\psi(s + \xi) - \mathcal{D}_\alpha(s + \xi, D_\xi) \sum_{|\beta| < |\alpha| \leq k} \xi^\beta \chi_{i\beta}(s) \mathcal{D}_\beta(s, D_s)\psi(s) \right| \leq$$

$$\leq C_k |\xi|^{k+1-|\alpha|} \sup_{|i| \leq k+1} \sup_{|s' - s| \leq |\xi|} |D^i \psi(s')| \rho^{-bk}(s, N_{\nu+1}), \quad |\alpha| = \deg \mathcal{D}_\alpha. \quad (3)$$

Outside the manifold N , where all operators \mathcal{D}_α are zero, this inequality is trivial; inside N it is substantive and is an analogue of Taylor's formula, in which the role of the operators D^i is played by the operators \mathcal{D}_α . In particular, if all polynomials p are identically zero, the system of operators \mathcal{D}_α coincides with the system of operators D^i , and (3) is Taylor's formula itself.

Lemma 3. For any infinitely differentiable function ψ , points s , $s + \xi \in N_\nu \setminus N_{\nu+1}$, an integer $k \geq 0$, and indices i and τ , the inequality holds

$$\left| G_\tau^i(s + \xi, D)\psi(s + \xi) - \frac{1}{i!} D_\xi^i \sum_{|j| \leq k} \xi^j G_\tau^j(s, D)\psi(s) \right|$$

$$\ll C_k |\xi|^{k+1-|i|} \sup_{|j| \leq k+1} \sup_{|s'-s| \leq |\xi|} |D^j \psi(s')| \rho^{-hk}(s, N_{\nu+1}).$$

Thus, on each identity $N_\nu \setminus N_{\nu+1}$, the quantities $G_\tau^i \psi$ may be regarded as derivatives of a certain infinitely differentiable function.

Using these lemmas, Whitney's extension theorem (see (7,3)) and the Seidenberg-Tarski principle (see (8)), we obtain solutions to our problems.

Theorem 1. *The ideal $\mathfrak{p}\mathcal{E}$ is the intersection of its local ideals, i.e. in order that a function ψ belong to $\mathfrak{p}\mathcal{E}$:*

$$\psi = \sum_{\tau} p_{\tau}(s) \varphi_{\tau}(s), \quad \varphi_{\tau}(s) \in \mathcal{E}, \quad (4)$$

it is necessary and sufficient that at each point s its formal Taylor series admit the expansion

$$\sum \frac{1}{i!} D^i \psi(s) \xi^i = \sum_{\tau} p_{\tau}(s + \xi) \varphi_{\tau,s}[\xi], \quad \varphi_{\tau,s}[\xi] \in \mathfrak{M}.$$

From this, in particular, follows the closedness of the ideal $\mathfrak{p}\mathcal{E}$. For some spaces \mathcal{E} , the assertion of the theorem in the case $\sigma = 1$, thanks to one theorem of Whitney (9), follows from (3,4). For arbitrary σ , this theorem was proved for some spaces by Malgrange (10).

Let us note one more consequence.

Corollary. If a function $\psi \in \mathcal{E}$ is divisible in this space by polynomials p and q , then it is divisible also by their least common multiple. More generally, if ψ belongs to the ideal generated by the polynomials p_1, \dots, p_{σ} and to the ideal generated by the polynomials q_1, \dots, q_t , then it belongs to the ideal generated by the least common multiples of all possible pairs p_i, q_i .

Theorem 1 asserts that in order for ψ to belong to $\mathfrak{p}\mathcal{E}$, it is necessary and sufficient that all functions $\mathcal{D}_{\alpha}(s, D)\psi(s)$ vanish identically; therefore, if we factor an arbitrary function $\psi \in \mathcal{E}$ by the ideal $\mathfrak{p}\mathcal{E}$, it is natural to expect that from it only the set of functions $\mathcal{D}_{\alpha}(s, D)\psi(s)$ will "remain." We shall now construct the space of such "remainders."

Definition. The space $\mathcal{E}_{\mathfrak{p}}$ is the space of elements Ψ of the following kind: Ψ is a collection of functions $\psi_{\alpha}(s)$, numbered at each point s by the same indices as the operators $\mathcal{D}_{\alpha}(s, D)$, continuous on each set $N_\nu \setminus N_{\nu+1}$ and satisfying the following conditions:

1. Each function $|\psi_{\alpha}(s)|$ is majorized by some function of the space \mathcal{E} .
2. For any points $s, s + \xi \in N_\nu \setminus N_{\nu+1}$ and integer $k \geq 0$, the inequality holds

$$\left| \psi_{\alpha}(s + \xi) - \mathcal{D}_{\alpha}(s + \xi, D_{\xi}) \sum_{|\beta| \leq |i| \leq k} \xi^i \mathcal{N}_{i\beta}(s) \psi_{\beta}(s) \right| \leq$$

$$\leq C_k |\xi|^{k+1-|\alpha|} \sup_{|\alpha| \leq k+1} \sup_{|s'-s| \leq |\xi|} |\psi_\alpha(s')| \rho^{-bk}(s, N_{\nu+1}).$$

3. Consider the operator R_ν^k , defined by induction by the equalities:

$$R_N^k \psi_\alpha = \psi_\alpha,$$

$$R_\nu^k \psi_\alpha(s) = R_{\nu'}^{r(k)} \psi_\alpha(s) -$$

$$- h_{\nu'}(s) \mathcal{D}_\alpha(s, D_\xi) \sum (s-s')^i \nu_{i\beta}(s') R_{\nu'}^{r(k)} \psi_\beta(s'), \quad \nu \leq N.$$

Here s' is the point of the manifold $N_{\nu+1}$ nearest to s ; ν' is the index of the manifold $N_{\nu'}$ to which s' belongs; $h_\nu(s)$ is an infinitely differentiable function equal to 1 inside the $1/2$ -neighborhood of N_ν and to zero outside the 1-neighborhood of this manifold; $r(k)$ is a certain fixed function of k , depending only on the polynomials \mathbf{p} .

The following relations hold:

$$R_\nu^k \psi_\alpha(s) = O(\rho^{k-|\alpha|}(s, N_{\nu+1}))$$

and, moreover, the functions

$$|R_\nu^k \psi_\alpha(s)| \rho^{k-|\alpha|}(s, N_{\nu+1})$$

are majorized by functions of the space \mathcal{E} .

The topology in the space \mathcal{E}_p is chosen to be the weakest topology under which it is complete.

Theorem 2. The mapping $\psi \rightarrow \Psi = \{\mathcal{D}_\alpha \psi\}$ establishes a topological isomorphism of the spaces $\mathcal{E}/\mathbf{p}\mathcal{E}$ and \mathcal{E}_p . •

Corollary. The general form of the solutions u of equation (1) with $\omega = 0$ in the space Φ of slowly increasing functions or functionals is given by the formula

$$\begin{aligned} (\tilde{u}, \varphi) &= \sum_{\nu \leq N, |\alpha| \leq q} \int_{N_\nu/N_{\nu+1}} \mathcal{D}_\alpha \varphi d\mu_{\nu,\alpha}(s) + \\ &+ \sum_{\nu \leq N, |\alpha| \leq q} \int_{N_\nu/N_{\nu+1}} R_\nu^q \mathcal{D}_\alpha \varphi d\lambda_{\nu,\alpha}(s), \end{aligned}$$

where $d\mu_{\nu,\alpha}$, $d\lambda_{\nu,\alpha}$ are arbitrary measures in R^n that are continuous functionals in Φ' , and $q \geq 0$ is an arbitrary integer.

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