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Abstract

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RADICALS IN GROUP PAIRS

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1. Let $\mathfrak{G}$ and Γ be two groups. Suppose, in addition, that an operation $\circ$ is given which assigns to each pair of elements $g \in \mathfrak{G}$ and $\sigma \in \Gamma$ an element of $\mathfrak{G}$, denoted by $g \circ \sigma$, and such that the mapping $g \to g \circ \sigma$ is an automorphism of the group $\mathfrak{G}$, and

$$g \circ (\sigma_1 \sigma_2) = (g \circ \sigma_1) \circ \sigma_2$$

for all $g \in \mathfrak{G}$ and $\sigma_1, \sigma_2 \in \Gamma$. The pair of groups $\mathfrak{G}$ and Γ, with respect to the operation $\circ$, forms a group pair $(\mathfrak{G}, \Gamma)$. Let $(\mathfrak{G}, \Gamma)$ be a group pair and let $\sigma \in \Gamma$. Denote by σ^f the automorphism of the group $\mathfrak{G}$ defined by the equality $\sigma^f(g) = g \circ \sigma$. The mapping $\sigma \to \sigma^f$ is a homomorphism of the group Γ into the group $A(\mathfrak{G})$ of all automorphisms of the group $\mathfrak{G}$, so that the operation $\circ$ defines a representation of the group Γ by automorphisms of the group $\mathfrak{G}$—a representation with respect to $\mathfrak{G}$. In defining a group pair one could likewise have started from a representation of Γ with respect to $\mathfrak{G}$. A group pair is called **faithful** if the kernel of the corresponding representation coincides with the identity of the group Γ. The operation $\circ$ will be called an **action**, Γ the **acting group**, and $\mathfrak{G}$ the **domain of action**.

A pair (H, Σ) is called a **subpair** of the pair $(\mathfrak{G}, \Gamma)$ if H and Σ are respectively subgroups in $\mathfrak{G}$ and Γ and if they form a group pair with respect to the action defined in $(\mathfrak{G}, \Gamma)$. In the obvious way one defines the notion of a local system of subpairs of a given group pair.

Let, further, $(\mathfrak{G}, \Gamma)$ and $(\bar{\mathfrak{G}}, \bar{\Gamma})$ be two group pairs, and let φ be a certain single-valued mapping of $\mathfrak{G}$ onto $\bar{\mathfrak{G}}$ and of Γ onto $\bar{\Gamma}$. The mapping φ is called a homomorphism if the mappings

$$\mathfrak{G} \xrightarrow{\varphi} \bar{\mathfrak{G}} \quad \text{and} \quad \Gamma \xrightarrow{\varphi} \bar{\Gamma}$$

are homomorphisms and if the relation

$$(g \circ \sigma)^{\varphi} = g^{\varphi} \circ \sigma^{\varphi}$$

holds for all $g \in \mathfrak{G}$ and $\sigma \in \Gamma$. If φ is a one-to-one mapping, then the homomorphism is an isomorphism of group pairs.

A property θ of a group pair is called **abstract** if, from the validity of θ for some pair $(\mathfrak{G}, \Gamma)$, it follows that θ is valid for every group pair isomorphic to $(\mathfrak{G}, \Gamma)$. If θ is an abstract property of group pairs, then $L\theta$ (locally θ) denotes the property of a pair to possess a local system of subpairs having the property θ. We shall call the pair $(\mathfrak{G}, \Gamma)$ **θ-triangular** if in $\mathfrak{G}$ there is an ascending normal series $[H_\alpha]$ of Γ-admissible subgroups (a Γ-series) such that all the induced pairs $(H_{\alpha+1}/H_\alpha, \Gamma)$ have property θ. If for the property θ one takes the property of a pair being trivial (which means that every element of Γ induces in $\mathfrak{G}$ the identity automorphism), then a θ-triangular pair with such θ is called a stable pair.

2. Let θ be an abstract property of group pairs, and let $(\mathfrak{G}, \Gamma)$ be a certain group pair. A **θ-subgroup** of the group Γ is any subgroup Σ of it such that the pair $(\mathfrak{G}, \Sigma)$ has property θ. Dually, θ-subgroups in $\mathfrak{G}$ are defined: a subgroup $H \subset \mathfrak{G}$ is called a **θ-subgroup** if H is Γ-admissible and the pair (H, Γ) has property θ. The **θ-radical** of the acting group Γ is the subgroup of Γ generated by all invariant θ-subgroups of Γ. Analogously, the θ-radical in the domain of action $\mathfrak{G}$ is defined.

We shall now present some results on θ-radicals of the group $\mathfrak{G}$.

Theorem 1. *The locally stable radical of the group $\mathfrak{G}$ is a locally stable subgroup.*

Indeed, let H be the locally stable radical of the group $\mathfrak{G}$. It is not hard to see that in H there is a Γ-series in each factor of which Γ induces a locally stable group of automorphisms. By Theorem 2.2 from [1] one may conclude that the pair (H, Γ) is locally stable, as required.

Next, let θ be an abstract group-theoretic property. We shall simultaneously regard θ, in the following way, as a property of group pairs. A pair $(\mathfrak{G}, \Gamma)$ has property θ if Γ induces in $\mathfrak{G}$ a group of automorphisms which is a θ-group. Under this interpretation of the property θ we shall denote it by $\tilde{\theta}$. $\tilde{\theta}$ is an abstract property of group pairs.

We shall conventionally say that a group-theoretic property θ satisfies condition (α) if: a) the complete direct product of θ-groups is again a θ-group; b) subgroups and homomorphic images of θ-groups are also θ-groups. If in condition a) the word "complete" is omitted, then we shall call the property θ a (β)-property.

Let $(\mathfrak{G}, \Gamma)$ be a group pair and let θ be some abstract group-theoretic property. An element $g \in \mathfrak{G}$ will be called a θ-element if the Γ-centralizer of the element g contains such a normal divisor Φ of the group Γ that Γ/Φ is a θ-group. In the case when θ is a (β)-property, the notion of a θ-element can also be defined in the following, more convenient way. Let A be a subgroup of some group B. By $\overline{A}$ we shall denote the intersection of all subgroups of the group B conjugate in B to A. If $(\mathfrak{G}, \Gamma)$ is a group pair and $g \in \mathfrak{G}$, then the properties of the group $\Gamma/\overline{\mathfrak{Z}_\Gamma(g)}^*$ characterize, to a certain extent, the actions of the whole group Γ on the element g. The element $g \in \mathfrak{G}$ is a θ-element relative to Γ if the group $\Gamma/\overline{\mathfrak{Z}_\Gamma(g)}$ is a θ-group.

The following theorem is easily proved.

Theorem 2. *If the property θ satisfies condition (α), then the set of all θ-elements of the group $\mathfrak{G}$ is a $\tilde{\theta}$-subgroup and contains the $\tilde{\theta}$-radical of the group $\mathfrak{G}$.*

The simplest example of an (α)-property is commutativity. An element $g \in \mathfrak{G}$ is called a **locally θ-element** (relative to Γ) if it is a θ-element relative to every subgroup of Γ having a finite number of generators.

Theorem 3. *Let the property θ be an abstract group-theoretic property satisfying condition (β), and let $(\mathfrak{G}, \Gamma)$ be an arbitrary group pair. Then the set of all locally θ-elements of $\mathfrak{G}$ is an $L\tilde{\theta}$-subgroup containing the $L\tilde{\theta}$-radical of the group $\mathfrak{G}$.*

Proof. Denote the set in question in the theorem by F. It is easily checked that F is a Γ-admissible subgroup. It is necessary to show that the pair (F, Γ) has property $L\tilde{\theta}$. We shall use the following notation. If A is a subgroup in $\mathfrak{G}$ and S is a subgroup in Γ, then by A^S we denote the minimal S-admissible subgroup in $\mathfrak{G}$ among those containing A. The pair (H, Σ) has a finite number of generators if Σ has a finite number of generators and if in H there is a subgroup with a finite number of generators A such that $A^\Sigma = H$. In an arbitrary group pair, the set of all subpairs with a finite number of generators forms a local system. Now let (H, Σ) be a subpair in (F, Γ) having a finite system of generators. We shall show that Σ induces in H a group of automorphisms which is a θ-group. Let $a_1, a_2, \dots, a_n$ be a finite set of elements of H such that $\{a_1, a_2, \dots, a_n\}^\Sigma = H$. Taking into account that

* $\mathfrak{Z}_\Gamma(g)$ is the Γ-centralizer of the element g, i.e. the set of all $\sigma \in \Gamma$ for which $g \circ \sigma = g$.

that the elements of H are θ-elements relative to Σ, choose, for each a_i in Σ, a normal divisor Φ_i of Σ, contained in $\mathfrak{Z}_\Sigma(a_i)$ and such that Σ/Φ_i is a θ-group. Let Φ be the intersection of all these Φ_i. Then Φ is a normal divisor in Σ, and Σ/Φ is a θ-group. Since Φ is a normal divisor in Σ, the Φ-center H (the totality of all elements $h \in H$ for which, for every $\sigma \in \Phi$, one has $h \circ \sigma = h$) is a Σ-admissible subgroup. Since this Φ-center contains all a_i, it coincides with H. It is now clear that the kernel of the representation of Σ relative to H contains Φ, and therefore Σ induces in H a group of automorphisms which is a θ-group. Thus (F, Γ) is an $L\tilde{\theta}$-pair. If (H, Γ) is a subpair in $(\mathfrak{G}, \Gamma)$ possessing the property $L\tilde{\theta}$, then every element of H is locally a θ-element. Consequently, $H \subset F$, which proves the theorem.

At the same time it has been proved that, under the conditions considered, the $L\tilde{\theta}$-radical of the group $\mathfrak{G}$ is itself an $L\tilde{\theta}$-subgroup.

Examples of (β)-properties are, for instance, finiteness and nilpotency of a group. If finiteness of a group is taken for θ, then $L\tilde{\theta}$ is equivalent to saying that the group Γ is locally periodic relative to $\mathfrak{G}$ in the sense of the definition in ([1]). If

θ is nilpotency of a group and the pair $(\mathfrak{G}, \Gamma)$ is an $L\tilde{\theta}$-pair, then we shall say that Γ is **generalized locally nilpotent relative to** $\mathfrak{G}$.

Starting from the θ-radicals of the group $\mathfrak{G}$, one can, just as in group theory, define the corresponding upper θ-radicals. In a number of cases the upper θ-radicals coincide with radicals defined by the property of θ-triangularity.

3. Here we shall consider some facts connected with the locally stable radical of the group Γ.

Let $(\mathfrak{G}, \Gamma)$ be a group pair. An element $\sigma \in \Gamma$ is called **locally stable** if it induces in $\mathfrak{G}$ a locally stable automorphism ([1],[2]).

Theorem 4. *If Γ is generalized locally nilpotent relative to $\mathfrak{G}$, then the totality of all locally stable elements of Γ is a locally stable invariant subgroup.*

Theorem 5. *Let $(\mathfrak{G}, \Gamma)$ be a group pair, and let the group $\mathfrak{G}$ be an LM-group (locally satisfies the maximality condition). The group Γ is locally stable if and only if it is generalized locally nilpotent relative to $\mathfrak{g}$ and is generated by locally stable elements.*

Whether in this theorem one can dispense with the restrictions on the group $\mathfrak{G}$ is unknown to us.

In the note ([2]), for the case of exact group pairs, the concept of the external radical of an acting group Γ was defined. We generalize this concept and define the external radical of the acting group of an arbitrary group pair. Let f be the homomorphism, defining the group pair $(\mathfrak{G}, \Gamma)$, of the group Γ into the group $A(\mathfrak{G})$, and let $R(\Gamma^f)$ be the abstract locally nilpotent radical of the group Γ^f. Denote by $R^f(\Gamma)$ the full inverse image in Γ of the radical $R(\Gamma^f)$. The **external radical of the group Γ relative to $\mathfrak{G}$** will mean the totality of all locally stable elements of $R^f(\Gamma)$. Since the group $R^f(\Gamma)$ is generalized locally nilpotent relative to $\mathfrak{G}$, the external radical Γ, which we shall denote by $R_{\mathfrak{G}}(\Gamma)$, is a locally stable normal divisor in Γ. It is easy to verify that if H is a Γ-admissible subgroup in $\mathfrak{G}$, then $R_{\mathfrak{G}}(\Gamma) \subset R_H(\Gamma)$.

A group pair $(\mathfrak{G}, \Gamma)$ is called **regular** if the normalizer of the subgroup Γ^f in the group $A(\mathfrak{G})$ contains the group of inner automorphisms $I(\mathfrak{G})$.

Theorem 6. *Let $(\mathfrak{G}, \Gamma)$ be a regular group pair, and let Z be the center of the group $\mathfrak{G}$. Then*

$$R_{\mathfrak{G}}(\Gamma) = R_Z(\Gamma) \cap R^f(\Gamma).$$

Proof. The inclusion $R_{\mathfrak{G}}(\Gamma) \leqslant R_Z(\Gamma) \cap R^f(\Gamma)$ is obvious. It remains to prove the reverse inclusion. We first prove that the pair $(\mathfrak{G}/Z, R^f(\Gamma))$ is locally stable. This pair is isomorphic to the pair $(I(\mathfrak{G}), R^f(\Gamma))$, defined by the action $x \circ \sigma = (\sigma^f)^{-1} x \sigma^f$ for all $x \in I(\mathfrak{G})$ and $\sigma \in R^f(\Gamma)$, so that it suffices to prove the local stability of the latter pair. Taking into account that $(R^f(\Gamma))^f = R(\Gamma^f)$, we see that the local stability of the pair $(I(\mathfrak{G}), R^f(\Gamma))$ is equivalent to the local stability of the pair $(I(\mathfrak{G}), R(\Gamma^f))$. From the hypotheses of the theorem it follows that the subgroup $R(\Gamma^f)$ is invariant relative to $I(\mathfrak{G})$. Noting that $R(\Gamma^f)$ is a locally nilpotent group, we now prove the local stability of the pair $(I(\mathfrak{G}), R(\Gamma^f))$ by referring to the following obvious proposition.

Let A and B be two normal divisors of some group C, and let B be locally nilpotent. Then B induces in A (by inner automorphisms) a locally stable group of automorphisms.

Thus it has been proved that the pair $(\mathfrak{G}/Z, R^f(\Gamma))$ is locally stable. Hence it follows that the group $R_Z(\Gamma) \cap R^f(\Gamma)$ is locally stable both relative to Z and relative to $\mathfrak{G}/Z$. By the already mentioned Theorem 2.2 of ([1]), this group is locally stable relative to $\mathfrak{G}$. We have proved the reverse inclusion, and consequently the theorem as well.

This theorem shows that the problem of finding the exterior radical of the group Γ relative to the group $\mathfrak{G}$, in the case of a regular pair, reduces to finding the exterior radical of Γ relative to the center of $\mathfrak{G}$. The following theorem for the locally stable radical has a similar character.

Theorem 7. *Let $(\mathfrak{G}, \Gamma)$ be a group pair and let $\mathfrak{G}$ be a nilpotent group. Then the locally stable radical of the group Γ coincides with the locally stable radical of Γ determined by the pair $(\mathfrak{G}/\mathfrak{G}', \Gamma)$.*

Here $\mathfrak{G}'$ denotes the commutator subgroup of the group $\mathfrak{G}$. The proof of the theorem follows directly from the results of the paper ([3]).

In note ([2]) it was proved that, if in the group pair $(\mathfrak{G}, \Gamma)$ the group $\mathfrak{G}$ satisfies the maximality condition, then the locally stable radical of the group Γ coincides with the intersection of all maximal locally stable subgroups of Γ. On the other hand, as V. G. Vilyatser showed, this will not always be the case. We now note the following theorem.

We shall say that the pair $(\mathfrak{G}, \Gamma)$ is an r-pair if the group $\mathfrak{G}$ has finite special rank.

Theorem 8. *Let $(\mathfrak{G}, \Gamma)$ be a group pair and let the subpair $(R(\mathfrak{G}), \Gamma)$ be a local r-pair. Then: a) the locally stable radical of the group Γ is a locally stable subgroup and coincides with the intersection of all maximal locally stable subgroups of Γ; b) a subgroup Σ of Γ is locally stable if and only if each element of Σ is a locally stable element.*

It is not yet known whether the locally stable radical of an acting group will always itself be a locally stable subgroup. In this connection we note that from Theorem 2.2 of ([1]) the following proposition follows directly:

Theorem 9. *Let $(\mathfrak{G}, \Gamma)$ be a group pair, and suppose that in the group $\mathfrak{G}$ there is a Γ-series $[H_\alpha]$ such that the locally stable radicals of Γ relative to all $H_{\alpha+1}/H_\alpha$ are locally stable. Then the locally stable radical of Γ relative to $\mathfrak{G}$ is a locally stable subgroup and coincides with the intersection of all locally stable radicals of Γ determined by the pairs $(H_{\alpha+1}/H_\alpha, \Gamma)$.*

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Note: Figure translations are in progress. See original paper for figures.