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Abstract

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AN EXAMPLE OF A DIMENSIONALLY DEFICIENT COMPACTUM

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In this note, for each prime number q a three-dimensional compactum $P_3(q)$ is constructed, for which $\dim(P_3(q) \times P_3(p)) = 4$ when $p \neq q$ and $\dim(P_3(q) \times P_3(q)) = 6$. In a certain sense this compactum generalizes the compactum $P_2(q)$ of L. S. Pontryagin (^{1,3}).

Definitions. Let a three-dimensional simplicial complex K be given in the Euclidean space R^7. A polyhedron T is called a **tube** in the complex K if the following conditions are satisfied: a) there exists a homeomorphism Φ mapping the product $E \times I$ of the disk E of radius 1 and the segment $I = [0, 1]$ onto the polyhedron T; b) the polyhedron T lies in $\widetilde{K}$ and does not intersect the one-dimensional skeleton of the complex K; c) if the intersection of T and a two-dimensional simplex t of the complex K is nonempty, then there exist numbers s_i, $i = 1, \ldots, n$, such that $0 \leq s_i \leq 1$ and

$$T \cap t = \bigcup_i \Phi(E \times s_i).$$

The sets $\Phi(E \times 0)$ and $\Phi(E \times 1)$ will be called the **bases** of the tube T. By definition, the tubes $T_1, \ldots, T_q$ have a **common base** if there exist numbers $\varepsilon_1, \ldots, \varepsilon_q$, equal to 0 or 1, such that the homeomorphisms $\varphi_i(x) = \Phi_i(x, \varepsilon_i)$ of the disk E coincide. We shall say that the tubes $T_1, \ldots, T_q$ **connect the two-dimensional complexes** L_1 and L_2 if these tubes have two common bases, one of which lies in L_1 and the other in L_2.

The compactum $P_3(q)$ will be obtained as the inverse limit of the spectrum of polyhedra $\{K_n, \mathfrak{D}_n^{n+1}\}$.

Construction of the complexes K_n and the projections $\mathfrak{D}_n^{n+1}$. Let K_0 be some triangulation of a three-dimensional simplex. Find pairwise disjoint tubes $T_1, \ldots, T_{s_1}$, whose bases lie on the boundary of the simplex K_0, such that every two-dimensional complex of the complex K_0 intersects at least one of the tubes $T_1, \ldots, T_{s_1}$. Let A be the open disk of radius 1/2, concentric with the disk E, and let S be its boundary. The homeomorphisms Φ_i could have been chosen so that the sets $\Phi_i(A \times I)$ are bodies of open subcomplexes of some subdivision of the complex K_0.

Define a mapping (identification) of the complex

$$K_0 \setminus \bigcup_i \Phi_i(A \times I)$$

in the following way:

I. $F_1(x) = F_1(y)$ if and only if $x = \Phi_i(a \times t)$, $y = \Phi_i(b \times t)$, where the points a and b belong to S and can be obtained from one another by a rotation of the circle S through an angle that is a multiple of $2\pi/q$. The subdivision of the complex K_0 could have been chosen so that the identification F_1 of the complex

$$K_0 \setminus \bigcup_i \Phi_i(A \times I)$$

was a simplicial mapping onto some complex K_1', which we shall assume embedded in Euclidean space.

Construct the mapping

II. $\mathfrak{D}_0^1 : K_1' \to K_0$.

Let $\mathfrak{D}_0^1(x) = F_1^{-1}(x)$, if $x \in \overline{\bigcup_i F_1(T_i)}$,

$$\mathfrak{D}_0^1(F_1(\Phi_i(a \times t))) = \Phi_i(b \times t),$$

where b is the point dividing the radius of the disk E, passing through ⋯

point a, in the same relation in which the point a divides the segment of this radius lying in the ring $E \setminus A$. Choose a sufficiently fine subdivision K_1' of the complex K_1 so that the diameters of the images of the simplices of the complex K_1 under the mapping $\mathfrak{D}_0^1$ are less than $1/2$. Denote by Q_i, $i = 1, \dots, s_1$, the subcomplexes $F_1(\Phi_i(S \times I))$, and by N_j the subcomplexes $F_1(t_j)$, $j = 1, \dots, r_1$, where t_j is an arbitrary closed simplex of a triangulation K_0.

To construct the complex K_2, find in the complex K_1 a system of tubes

$$T_{s_1+1}, \dots, T_{s_2},$$

satisfying the following conditions:

1. The base of each tube lies either on a two-dimensional simplex that is a face of only one three-dimensional simplex, or on a simplex from $\bigcup_i Q_i$. In the latter case there will be q tubes having this base in common.
2. Two tubes may intersect only in common bases.
3. Each two-dimensional simplex of the complex K_1 intersects at least one of the tubes $T_{s_1+1}, \dots, T_{s_2}$.

4. Any two two-dimensional simplices lying in the subcomplex

   $$\bigcup_{i=1}^{s_1} Q_i \cap N_j$$

   are joined by q tubes in the subcomplex N_j.

If the system of tubes satisfies these conditions, then the subcomplex

$$\bigcup_{s_1+1}^{s_2} \Phi_i(A \times I)$$

of some subdivision of the complex K_1 will be open. The identification I, applied to the complex K_1, defines a simplicial mapping F_2 of the complex

$$K_1 \setminus \bigcup_{s_1+1}^{s_2} \Phi_i(A \times I)$$

onto a certain complex K_2', which we shall assume embedded in Euclidean space R^7. Formulae II define a mapping

$$\mathfrak{D}_1^2 : K_2' \to K_1.$$

Choose such a subdivision K_2 of the complex K_2' that the diameters of the images of the three-dimensional simplices of the complex K_2 under the mappings $\mathfrak{D}_1^2$ and $\mathfrak{D}_0^2 = \mathfrak{D}_1 \mathfrak{D}_1^2$ are less than $1/2^2$.

III. Let

$$Q_i = F_2(Q_i)$$

for $i \leq s_1$, and

$$Q_i = F_2(\Phi_i(S \times I))$$

for $s_1 + 1 \leq i \leq s_2$. Let $N_j = F_2(N_j)$ for $j \leq r_1$, and $N_j = F_2(t_j)$ for $r_1 + 1 \leq j \leq r_2$ (t_j is an arbitrary closed three-dimensional simplex of the complex K_1).

Suppose that the complexes K_i, the mappings $\mathfrak{D}_{i-1}^i$ for $i \leq n$, and the subcomplexes Q_k and N_l for $k \leq s_n$ and $l \leq r_n$ have already been constructed. Find in the complex K_n a system of tubes

$$T_{s_n+1}, \dots, T_{s_{n+1}},$$

satisfying conditions 1–3 and the following condition:

4′. For $s_i < j \leq s_{i+1}$, any two two-dimensional simplices of the complex

$$N_j \cap \bigcup_{k=s_i+1}^{s_n} P_k$$

are joined by q tubes in the complex N_j. Just as in I and II, we define a mapping F_{n+1} of the complex

$$K_n \setminus \bigcup_{s_n+1}^{s_{n+1}} \Phi_i(A \times I)$$

onto a complex K'_{n+1}, and a mapping

$$\mathfrak{D}_n^{n+1} : K'_{n+1} \to K_n.$$

Let K_{n+1} be a subdivision of the complex K'_{n+1} for which the diameters of the images of the three-dimensional simplices under the mappings

$$\mathfrak{D}_i^{n+1} = \mathfrak{D}_i^{i+1} \dots \mathfrak{D}_n^{n+1}$$

are less than $1/2^{n+1}$. As in III, we define the system of subcomplexes Q_i, N_j for $i \leq s_{n+1}$, $j \leq r_{n+1}$.

We denote by $P_3(q)$ the inverse-limit space of the spectrum

$$\{K_n, \mathfrak{D}_n^{n+1}\}.$$

Remark. The construction of the compactum $P_3(q)$ is not unique, so that we have constructed not a single compactum, but a class of compacta $P_3(q)$. The compacta $\mathfrak{D}_n^{-1} N_j$ also belong to the class of compacta $P_3(q)$.

Computation of the cohomological dimensions of the compacta $P_3(q)$. The coefficient domains are as follows: Z is the group of integers; Z_p is the group of residues modulo p; Q is the group of rational numbers; Q_p is the group of rational numbers of the form a/p^α, reduced modulo 1; R_p is the group of rational numbers whose denominator is not divisible by p.

Let $P = P_3(q)$, $\dot{P} = \tilde{\omega}_0^{-1}(\dot{K}_0)$, and $\dot{K}_n = (\tilde{\omega}_0^n)^{-1} \dot{K}_0$, where $\dot{K}_0$ is the boundary of the simplex K. It is easy to see that $H^3(P, \dot{P}; Z) \approx Z_q$. Hence (2), $\dim P = cd_Z P = 3$. To compute the Čech homology group $H_2(P, \dot{P}; G)$, the following lemma is needed.

Lemma. Any two-dimensional cycle z of the complex $(K_n, \dot{K}_n)$ is homologous in $(K_n, \dot{K}_n)$ to a cycle lying on the subcomplex

$$M_n = \bigcup_{i=1}^{s_n} Q_i.$$

Proof. Suppose the lemma has been proved for the complexes K_i with $i \leq n$. We prove it for the complex K_{n+1}. Let

$$L = \bigcup_{i=s_n+1}^{s_{n+1}} \Phi_i(A \times I),$$

$\overline{L}$ be the closure of L, and $\dot{L}$ the boundary of L. Represent the cycle z as a sum of chains

$$z = z_1 + z_2,$$

where z_1 lies in $K_{n+1} \setminus F_{n+1}(\dot{L})$ and z_2 in $F_{n+1}(\dot{L})$. Since the mapping F_{n+1} onto $K_n \setminus \overline{L}$ was a homeomorphism, there is a chain x_1 in $K_n \setminus \overline{L}$ for which $F_{n+1}(x_1) = z_1$. Then

$$F\Delta x_1 = \Delta F x_1 = -\Delta z_2.$$

It is not difficult to see that any one-dimensional cycle z in $\dot{L}$ such that $F_{n+1}(z) \sim 0$ in $F_{n+1}(\dot{L})$ is itself homologous to zero in $\dot{L}$. Thus there exists a chain y_1 in $\dot{L}$ for which

$$\Delta x_1 = \Delta y_1.$$

The cycle $x_1 - y_1$ is homologous in K_n to some cycle y_2 from M_n, i.e. there exists a chain X^3 for which

$$\Delta X^3 = x_1 - y_1 - y_2.$$

Represent the chain X^3 as a sum of chains:

$$X^3 = X_1^3 + X_2^3,$$

where X_1^3 lies on $K_n \setminus \overline{L}$ and X_2^3 on $\overline{L}$. The chain

$$\Delta X_2^3 + y_1 + y_2$$

lies in $M_n \cup \overline{L}$, and

$$F x_1 - F\Delta X_1^3 = F(\Delta X_2^3 + y_1 + y_2).$$

Therefore

$$z - \left(z_2 + F(\Delta X_2^3 + y_1 + y_2)\right) = \Delta F X_1^3.$$

This is the homology we need. The lemma is proved.

We now prove that

$$H_2(P, \dot{P}; G) = 0$$

if the group G admits unique division by q. Indeed,

$$H_2(P, \dot{P}; G) \approx \varinjlim \{H_2(K_n, \dot{K}_n; G); (\tilde{\omega}_n^{n+1})_*\}.$$

Let

$$\alpha = \{a_n\} \in \varinjlim \{H_2(K_n, \dot{K}_n; G); (\tilde{\omega}_n^{n+1})_*\}.$$

The element a_{n+1} contains a cycle z_{n+1} lying on the subcomplex M_{n+1}. This cycle takes equal values on all simplices of the subcomplex

$$\bigcup_{i=1}^{s_n} Q_i.$$

Indeed, any two simplices τ_i and τ_j of this subcomplex in K_{n+1} are joined by q tubes $Q_{l_1}, \dots, Q_{l_k}$, $s_n < l_m \leq s_{n+1}$. Computing the value of the boundary of the cycle z_{n+1} on the common bases of these tubes, we obtain

$$q \cdot z_{n+1}(\tau_i) = \sum_{m=1}^{q} z_{n+1}(Q_{l_m}) = q z_{n+1}(\tau_j)$$

(the cycle z_{n+1} takes equal values on all simplices of the subcomplex Q_{l_m}). But the cycle $\tilde{\omega}_n^{n+1} z_{n+1}$ is homologous to a cycle z_n taking, on the simplices of Q_i for $i \leq s_n$, the same values as the cycle z_{n+1} (hence values equal to one another). Therefore the cycle

$$\tilde{\omega}_n^{n+1}(z_{n+1}) \sim 0$$

in K_n, $a_n = 0$, and $\alpha = 0$.

Theorem. Let $cd_G X$ be the cohomological dimension of a compactum X over the coefficient domain G. Then

$$cd_Z P_3(q) = cd_{R_q} P_3(q) = cd_{Z_q} P_3(q) = 3,$$

$$cd_{Q_q} P_3(q) = 2, \qquad cd_{Z_p} P_3(q) = cd_{R_p} P_3(q) = cd_{QP} 3(q) = 1 \quad \text{for } p \neq q.$$

We shall prove only that $cd_{Z_p} P_3(q) = 1$ (the remaining assertions are proved analogously).

Let $cd_{Z_p} P_3(q) = l$, $l > 1$. Then the compactum $P_3(q)$ contains an l-dimensional subcompactum F, each point x of which has the following property: there is a neighborhood U of the point x such that for any neighborhood $x \in V \subset U$

the group $H^l(\bar{V}, \dot{V}; Z_p)$ is nontrivial (7). Let K_n^2 be the two-dimensional skeleton of the complex K_n. The subspace $\bigcup_{i=0}^{\infty} \omega_i^{-1} K_i^2$ of the compactum $P_3(q)$ is the sum of a countable number of compacta homeomorphic to L. S. Pontryagin' s compactum $P_2(q)$. Since $\mathrm{cd}_{Z_p} P_2(q) = 1$, by the sum theorem for cohomological dimension (6) the compactum F contains a point x not belonging to $\bigcup_{i=0}^{\infty} \omega_i^{-1} K_i^2$.

For any neighborhood U of the point x there exist a number n and an open three-dimensional simplex t of the complex K_n such that $x \in \omega_n^{-1} t \subset U$. It is not difficult to prove that the closure of the neighborhood $\omega_n^{-1} t$ coincides with $\omega_n^{-1}(\bar{t})$, and its boundary with $\omega_n^{-1}(\dot{t})$. But the compactum $\omega_n^{-1}(\bar{t})$ belongs to the class of compacta $P_3(q)$. Thus $H^i(\omega_n^{-1}(\bar{t}), \omega_n^{-1}(\dot{t}); Z_p) = 0$ for $i = 2, 3$. Therefore $l < 2$ and $\mathrm{cd}_{Z_p} P_3(q) = 1$. The theorem is proved.

From Bokshtein' s formulas (4,5), expressing the dimension of a product of compacta in terms of the cohomological dimensions of the factors computed by us, it follows that $\dim(P_3(q) \times P_3(p)) = 4$ for $p \neq q$ and $\dim(P_3(q) \times P_3(q)) = 6$.

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Note: Figure translations are in progress. See original paper for figures.