MATHEMATICS

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Abstract

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MATHEMATICS

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ON THE ABSOLUTE MINIMUM OF FUNCTIONALS ON A SET OF FUNCTIONS WITH BOUNDED DERIVATIVE

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The problem of the absolute minimum of the functional

$$I(u) = \int_a^b F(x, y, p)\, dx \tag{1}$$

is considered on the set U^p of piecewise-smooth curves satisfying the conditions:

$$y' = g(x, y, p); \quad |p| \leqslant 1; \quad y(a) = a_1, \quad y(b) = b_1. \tag{2}$$

The functions $F(x, y, p)$ and $g(x, y, p)$ are continuous together with their partial derivatives with respect to all three arguments for arbitrary x, y and for $|p| \leqslant 1$. Moreover,

$$\frac{\partial g}{\partial p} > 0 \tag{3}$$

and the functions $g(x, y, \pm 1)$ have constant sign.

It follows from (2) that any curve $u \in U^p$ belongs to a closed simply connected domain G of the x, y-plane, whose upper boundary $y = \Gamma_2(x)$ consists of solutions of the equations

$$y' = g(x, y, 1); \quad y' = g(x, y, -1), \tag{4}$$

passing respectively through the points $A(a, a_1)$ and $B(b, b_1)$, and whose lower boundary $y = \Gamma_1(x)$ consists of the solution of the first equation (4) passing through B, and of the second equation (4) passing through A. Let

$$y = \varphi(x, \tau); \quad y = \psi(x, t) \tag{5}$$

be the solutions of the first and second equations (4), respectively. Here τ and t are integration constants. Along any curve $u \in U^p$, the equation $y = \varphi(x, \tau)$ defines a piecewise-smooth function $x = x(\tau)$, having discontinuities of the first kind at the values $\tau = \mu_i$ $(i = 1, 2, \dots, n)$ corresponding to portions $p(x) \equiv 1$ of the curve u. The parameter τ can be chosen so that $x(\tau)$ increases. Making in (1) and (2) a change of independent variable, we shall have

$$I(u) = \sum_{i=1}^{n} \int_{\mu_i+0}^{\mu_{i+1}-0} F_1(\tau, y, p)\, d\tau + \sum_{i=1}^{n} \Phi(\mu_i, y_i, \bar{y}_i) \tag{6}$$

under the condition

$$\dot{y} \equiv \frac{dy}{dt} = g_1(\tau, y, p); \quad |p| \leqslant 1; \quad \mu_1 \equiv \alpha = \tau(a, a_1); \quad \mu_n \equiv \beta = \tau(b, b_1). \tag{7}$$

Here:

$$F_1 = F(\varphi_1(\tau, y), y, p) \frac{\varphi_{1\tau} g(\varphi_1, y, 1)}{g(\varphi_1, y, 1) - g(\varphi_1, y, p)}; \tag{8}$$

$$g_1 = g(\varphi_1, y, p) \frac{\varphi_{1\tau} g(\varphi_1, y, 1)}{g(\varphi_1, y, 1) - g(\varphi_1, y, p)}; \tag{9}$$

$$\Phi(\tau, y, \bar{y}) = \int_{\bar{x}}^{x} F(x, \varphi(x, \tau), 1)\, dx = \int_{\bar{y}}^{y} \frac{F(\varphi_1(\tau, \eta), \eta, 1)}{g(\varphi_1(\tau, \eta), \eta, 1)}\, d\eta; \tag{10}$$

$$y = y(\tau + 0); \qquad \bar{y} = y(\tau - 0); \tag{11}$$

$x = \varphi_1(\tau, y)$ is the inverse function to $y = \varphi(\tau, x)$.

From equation (7) one can express $p(\tau, y, \dot{y})$. Substituting in (8), we obtain the integrand in the form $\widetilde{F}_1(\tau, y, \dot{y})$.

We have

$$W(\tau, y) = \lim_{\dot{y} \to \infty} F_1(\tau, y, \dot{y}) \cdot \frac{1}{\dot{y}} = \frac{F(\varphi_1, y, 1)}{g(\varphi_1, y, 1)}. \tag{12}$$

Let us introduce for consideration the set U^τ of piecewise-smooth lines on which $y(\tau)$ is single-valued everywhere except for a finite number of points μ_i, where $y(\tau)$ has discontinuities of the first kind. We have: $U^p \subset U^\tau$. The functional $I(u)$, $u \in U^\tau$, is defined by formula (6). In view of (12), this definition corresponds to definition (4) from (1), with the case

$$W(x,y,1) = W(x,y,-1).$$

Let us introduce for consideration the set of (y,z)-lines U_0^τ, where $y(\tau)$ is a function of zero proximity to the line $u_0 \in U_0^\tau$, $z(\tau)$ is its local slope in the coordinates τ, y (see (1), definition 3), and the function

$$S(\tau,y,z) = \widetilde{F}_1(\tau,y,z) - W(\tau,y)z - \int_c^y W_\tau(\tau,\eta)\,d\eta; \tag{13}$$

c is an arbitrary constant. We shall call the local value of the parameter p the quantity q defined by the equation

$$z = g_1(\tau,y,q). \tag{14}$$

By virtue of the properties of the function $g_1(\tau,y,q)$, a line $u \in U_0^\tau$ can be specified either by the pair (y,z) or by the pair (y,q). Substituting (14) in (13), we obtain:

$$S = F_1(\tau,y,q) - F_1(\tau,y,1)\frac{g(\tau,y,q)}{g(\tau,y,1)} - \int_c^y \left[\frac{F(\varphi_1(\tau,\eta),\eta,1)}{g(\varphi_1(\tau,\eta),\eta,1)}\right]_\tau d\eta. \tag{15}$$

The following is easily proved:

Lemma. Let there be a (y,q)-line $u_0 \in U_0^\tau$. In order that there exist a sequence of broken lines $\{\gamma_n\} \to u_0$, $\{\gamma_n\} \subset U^p$, it is necessary and sufficient that the functions $y(\tau)$ and $q(\tau)$ satisfy the conditions

$$q(\tau) \geq -1; \qquad \dot{y} - g_1(\tau,y,q) \geq 0; \tag{16}$$

at the points of continuity of $y(\tau)$, and the condition

$$y(\mu_i) - \bar{y}(\mu_i) \geq 0 \tag{17}$$

at the points of discontinuity of $y(\tau)$.

Let $\tilde{u} \in U_0^\tau$ be a (y,q)-line satisfying the condition

$$S(\tau,\tilde{y},\tilde{q}) = \inf S(\tau,y,q), \qquad \Gamma_1(\tau) \leq y \leq \Gamma_2(\tau); \quad |q| \leq 1 \tag{18}$$

for each fixed $\tau \in (\alpha,\beta)$. From Theorem 4 ([1]) and the lemma it follows:

Theorem 1. Let the (y,q)-line $\tilde{u} \in U_0^\tau$ satisfy conditions (16), (17), and (18). Then $\tilde{u}$ is an absolute minimum of the functional (1), i.e.

$$I(\tilde{u}) = \inf_{u \in U^p} I(u), \tag{19}$$

where

$$I(\tilde{u}) = \int_{\alpha}^{\beta} S(\tau, \tilde{y}, \tilde{q})\, d\tau + \Phi(\beta, b_1) - \Phi(\alpha, a_1). \tag{20}$$

Theorem 2. Let $[\tau_1, \tau_2] \in [\alpha, \beta]$ be an isolated interval on which the functions $\tilde{y}(\tau)$, $\tilde{q}(\tau)$, specified by (18), satisfy (16) and (17). Construct the following arc v: 1) on $[\alpha, \tau_1]$, v coincides with the absolute minimum v_1 of the functional

$$I_1(u_1, y_1) = \int_{\alpha}^{\tau_1} F_1(\tau, y, p)\, d\tau - \Phi(\tau_1, y_1) \tag{21}$$

with movable right end y_1; 2) on $[\tau_1, \tau_2]$, v is a (y, q)-line: $y = \tilde{y}(\tau)$, $q = \tilde{q}(\tau)$; 3) on $[\tau_2, \beta]$, v coincides with the absolute minimum v_2 of the functional

$$I(u_2, y_2) = \int_{\tau_2}^{\beta} F_1(\tau, y, p)\, d\tau + \Phi(\tau_2, y_2) \tag{22}$$

with movable left end y_2. If

$$y_1 \leq \tilde{y}(\tau_1); \qquad y_2 \geq \tilde{y}(\tau_2), \tag{23}$$

then

$$I(v) = \inf I(u), \qquad u \in U^p. \tag{24}$$

Proof. Along any line $u \in U^p$ we have:

$$I(u) = \int_{\alpha}^{\tau_1} F_1\, d\tau + \int_{\tau_1}^{\tau_2} S(\tau, y, q)\, d\tau + \Phi(\tau_2, y_2) - \Phi(\tau_1, y_1) + \int_{\tau_2}^{\beta} F_1\, d\tau =$$

$$= I_1(u_1, y_1) + I_2(u_2, y_2) + \int_{\tau_1}^{\tau_2} S(\tau, y, p)\, d\tau;$$

according to the hypothesis of the theorem,

$$I(v) = I_1(v_1) + I_2(v_2) + \int_{\tau_1}^{\tau_2} S(\tau, \tilde{y}, \tilde{q})\, d\tau \leq I(u), \qquad u \in U^p.$$

On the other hand, from the hypothesis of the theorem and from the lemma it follows that there exists a sequence $\{\gamma_n\} \subset U^p$ such that $I(\gamma_n) \to I(v)$. Therefore, by the definition of the lower bound, (24) is valid.

Obviously, analogous theorems also hold in the t, y-plane (with the sign of the inequalities (16), (17), (23) changed).

It follows from Theorem 1 that functionals on the set of functions with bounded derivative have (y, q)-minimals with a "split" derivative, analogous to the (y, z)-minimals of the simplest functional ([1,3]). The role that vertical directions played for the latter is played, for the present minimals, by the directions $p = \pm 1$.

The solution of the minimum problem (1) should begin not with the formulation and solution of the Euler equation, but with the solution of the elementary problem of minimizing a function of two variables $S(\tau, y, q)$ for fixed $\tau \in (\alpha, \beta)$. If the conditions of Theorem 1, or its analogue in the plane t, y, turn out to be satisfied, then the solution ends there.

If, a posteriori, it turns out that $y' \equiv g(x, y, q)$, then the minimal $u \in \overline{U}^p$. Otherwise $\bar{u} \in U_0^\sigma$ or $\bar{u} \in U_0^t$, and the pair $(y, \bar{q})$ determines a minimizing sequence $\{r_n\} \subset U^p$, analogous to $\{r_n\} \subset U$ in ([1,2]).

Theorem 2 makes it possible to single out (y, q)-segments of a minimal if the conditions of Theorem 1 are not satisfied.

If the conditions of Theorems 1 and 2 and of their analogues are not satisfied, then the minimal may turn out to be piecewise smooth, and it can be found by the usual classical methods.

The results presented have interesting applications in mechanics.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.