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# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# DIVERGENT SERIES WITH RESPECT TO THE HAAR SYSTEM AND BASES

*(Presented by Academician P. S. Novikov on 21 I 1961)*

§ 1. This note is a direct continuation of the investigations that we published in the article <sup>(12)</sup>.

In the work of A. N. Kolmogorov and D. E. Menshov <sup>(3)</sup> a statement was formulated (belonging to A. N. Kolmogorov) which says:

*There exists a function  $f(x) \in L^2(0, 2\pi)$  such that the terms of its trigonometric Fourier series can be rearranged so that the newly obtained series diverges almost everywhere on  $[0, 2\pi]$ .*

Starting from this result, in <sup>(10)</sup> we established that the following is true.

**Theorem I.** *There exists a function  $F(x) \in L^p(0, 2\pi)$  for all  $p > 0$ , and such that, for any summability method  $T^*$ , the terms of the trigonometric Fourier series of the function  $F(x)$  can be rearranged so that the newly obtained series is not summable by the method  $T^*$  almost everywhere on  $[0, 2\pi]$ .*

In connection with Theorem I it is useful to note that in it one can assert not only nonsummability by the method  $T^*$ , but even the unboundedness of the  $T^*$ -means of certain rearranged trigonometric Fourier series of the function  $F(x)$ . Moreover, Theorem I is also valid for summability methods that are applied directly to series (and not to sequences). Such methods we shall denote by  $\gamma^* = \|b_{n,m}\|$ , and we shall only assume that  $\lim_{n \rightarrow \infty} b_{n,m} = 1$  ( $m = 1, 2, \dots$ ).

In Zagorskii' s work <sup>(9)</sup>, although without a detailed proof, a scheme is given for constructing an almost everywhere divergent trigonometric series of class  $L^2$ . Modifying Zagorskii' s construction <sup>(9)</sup> and using one of our assertions (see <sup>(10)</sup>, p. 816), we announced in <sup>(12)</sup> a result which says that Theorem I remains valid also for the Walsh system. In addition, in the same work <sup>(12)</sup> the following was formulated:

**Theorem II.** *There exists a function  $F(x) \in L^2(0, 1)$  such that, for every summability method  $T^*$ , some rearranged Fourier series of the function  $F(x)$  with respect to the Haar system  $\{\chi_n^{(h)}(x)\}$  is not summable by the method  $T^*$  almost everywhere on  $[0, 1]$ .*

§ 2. In this paragraph we shall present new results concerning series with respect to the Haar system. First we introduce the following.

**Definition 1.** Let  $\psi_i(x)$  ( $i = 1, 2, \dots$ ) be measurable functions on  $[0, 1]$ , finite almost everywhere. We then say that the series

$$\sum_{i=1}^{\infty} c_i \psi_i(x) \quad (c_i \text{ are certain constants}) \quad (1)$$

has property  $D$ , if for every summability method  $\gamma^* = \|b_{n,m}\|$  (method  $T^*$ ) the terms of the series (1) can be rearranged so that the newly ...

the resulting series was not summable by the method  $\gamma^*$  (by the method  $T^*$ ) at almost every point  $x \in [0, 1]$ . Moreover, if the method  $\gamma^*$  satisfies the condition

$$\lim_{m \rightarrow \infty} b_{n,m} = 0 \quad (n = 1, 2, \dots),$$

then the terms of the series (1) can be rearranged so that the  $\gamma^*$ -means ( $T^*$ -means) are unbounded at almost every point  $x \in [0, 1]$ .

**Theorem 1.** Whatever increasing sequence of integers  $n_m$  is given, there exists a series

$$\sum_{m=1}^{\infty} A_m(x), \quad \text{with} \quad A_m(x) = \sum_{s=1}^{2^{n_m}} a_{n_m}^{(s)} \chi_{n_m}^{(s)}(x)$$

and

$$\sum_{m=1}^{\infty} \sum_{s=1}^{2^{n_m}} (a_{n_m}^{(s)})^2 < \infty$$

such that, after a certain rearrangement of the functions  $A_m(x)$ , the series

$$\sum_{\nu=1}^{\infty} A_{m_\nu}(x)$$

diverges unboundedly almost everywhere on  $[0, 1]$ .

From Theorem 1 we see that, in order to construct divergent Fourier series of class  $L^2$  with respect to a rearranged Haar system, it is not necessary to take all Haar functions. Namely, it is enough to take only those Haar functions for which the lower index assumes infinitely many values  $n_i$ , while the upper index then runs through  $1 \leq s \leq n_i$ .

**Theorem 2.** There exists a function  $F(x) \in L^p(0, 1)$  for all  $p > 0$  such that, if

$$\sum_{n=1}^{\infty} \sum_{s=1}^{2^n} a_n^{(s)} \chi_n^{(s)}(x) = \sum_{m=1}^{\infty} a_m \chi_m(x) \quad (2)$$

is the Fourier series of the function  $F(x)$ , then the series (2) has property  $D$ .

**Theorem 3.** Let  $\omega(n) \uparrow \infty$  and  $\omega(n) = o(\log n)$ . Then there exists a function  $f(x) \in L^2(0, 1)$  whose Fourier series  $\sum a_m \chi_m(x)$ , after a certain rearrangement of the terms  $\sum a_{p_\nu} \chi_{p_\nu}(x)$ , diverges almost everywhere on  $[0, 1]$ , and nevertheless

$$\sum a_m^2 \omega(m) < \infty, \quad \sum a_{p_\nu}^2 \omega(p_\nu) < \infty.$$

Here  $\chi_m(x) = \chi_n^{(k)}(x)$  if  $m = 2^n + k$  with  $1 \leq k \leq 2^n$ .

**Theorem 4.** For every  $\varepsilon > 0$ , the sequence  $(\log n)^{1+\varepsilon}$  is both a Weyl multiplier for unconditional convergence almost everywhere of series with respect to the Haar system  $\{\chi_m(x)\}$ , and a Weyl convergence multiplier for series with respect to any rearranged Haar system.

The special interest of Theorems 3 and 4 is due (in our opinion) to the fact that, in particular, they give the approximate order of the Weyl multiplier for unconditional convergence almost everywhere of series with respect to the Haar system. This is the first result concerning classical orthonormal systems in which the question is one of the sharpness of the Weyl multiplier for unconditional convergence. As for trigonometric series (as well as series with respect to the Walsh system), in the indicated direction there are no more or less definitive results.

§ 3. In this section we shall state results concerning series with respect to bases. We first recall:

**Definition 2.** Let  $E$  be a space of type  $B$  (Banach). Then a system  $f_i \in E$  ( $i = 1, 2, \dots$ ) is called a **basis** in  $E$  if, for every  $f \in E$ , there exists a unique numerical sequence  $\{c_i\}$  such that

$$f = \sum c_i f_i.$$

**Definition 3.** Let  $E = L^2 = L^2(0, 1)$ , and let  $f_i = f_i(x)$  be a basis in the space  $L^2(0, 1)$ . Then the system  $\{f_i(x)\}$  is called a **Riesz basis** if, for every  $f \in L^2$  with  $f = \sum c_i f_i$ , the inequality holds:

$\sum c_i^2 < \infty$  and, moreover, if for every sequence  $\{d_i\}$  with  $\sum d_i^2 < \infty$  the series  $\sum d_i f_i$  is the expansion of some  $F \in L^2$ .

The last definition was introduced by N. K. Bari (<sup>2</sup>).

In the definitions indicated the following are true:

**Theorem 5.** Let  $\{f_k(x)\}$  be a basis in  $L^2(0, 1)$ . Then there exists a function  $F(x) \in L^2(0, 1)$  such that the series  $\sum_{k=1}^{\infty} \alpha_k f_k(x)$ , which is the expansion (in the  $L^2$ -norm) of the function  $F(x)$  with respect to the basis  $\{f_k(x)\}$ , has property  $D$ .

In proving Theorem 5 we used the following considerations: a) Theorem 1 of §1; b) Banach's theorem (<sup>1</sup>, pp. 95-96) stating that if  $\{f_i\}$  is a basis in a space  $E$  of type  $B$ , then there is a sequence of linear continuous functionals  $\{U_i(x)\}$ ,

defined on  $E$ , such that  $\{U_i, x_i\}$  is a biorthogonal system; c) the method of proof by Marcinkiewicz <sup>(6)</sup> (see also <sup>(4)</sup>, pp. 352-357) of the theorem that every orthonormal system has a convergence subsequence (this assertion was proved by Marcinkiewicz <sup>(6)</sup> and Menshov <sup>(7)</sup>); d) one of our assertions concerning the summation of divergent series (<sup>(11)</sup>, Corollary 3, p. 792).

**Theorem 6.** *Let  $\{f_i(x)\}$  be a Riesz basis in the space  $L^2(0, 1)$ , and suppose it is uniformly bounded. Then there exists a function  $F(x) \in L^p(0, 1)$  for all  $p > 0$  such that the series  $\sum_{k=1}^{\infty} \alpha_k f_k(x) = F(x)$  converges almost everywhere on  $[0, 1]$  and has property D.*

§ 4. In a review article on Cooke's book I formulated a problem (see <sup>(5)</sup>, p. 452, Problem 3)\*\*:

*Does there exist an orthonormal complete system  $\{\varphi_n(x)\}$  that is a system of unconditional convergence (i.e., every series  $\sum a_n \varphi_n(x)$  ( $x \in [0, 1]$ ), in any order of its terms, converges almost everywhere on  $[0, 1]$ , as soon as  $\sum a_n^2 < \infty$ )?*

We shall give an answer\*\*\* to this question, and shall also prove several other theorems concerning orthonormal complete systems.

**Theorem 7.** *Let  $\{\varphi_n(x)\}$  be an orthonormal complete system on the interval  $[0, 1]$ . Then there exists a series  $\sum a_n \varphi_n(x)$  with  $\sum a_n^2 < \infty$  which converges almost everywhere on  $[0, 1]$  and has property D.*

**Remark.** Although Theorem 7 in fact follows from Theorem 5, nevertheless we have formulated it separately, since it seems to us to have fundamental significance. In general, it may be useful to note that Theorem 7 follows (in its essence) from Theorem 1, if one takes into account the idea of Marcinkiewicz's arguments. As for Theorem 5, it also follows (in essence) from Theorem 1 if, in addition to Marcinkiewicz's arguments, one also uses Banach's theorem.

From Theorem 7 it follows immediately that the following is true:

**Theorem 8.** *There does not exist an orthonormal complete system  $\{\varphi_n(x)\}$  that would be a system of unconditional convergence almost everywhere.*

This assertion gives a negative answer to the problem stated above.

From Theorem 7, taking into account Kaczmarsz's theorem (<sup>(4)</sup>, pp. 205-207), it also follows:

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\* Here equality is understood in the sense of convergence (in the  $L^2$ -norm) of the series on the left to  $F(x)$ .

\*\* We do not know whether the statement of such a problem had been published anywhere earlier.

\*\*\* Results in the same direction have been obtained by A. M. Olevskii <sup>(8)</sup>, whose work is being published in the present issue.

**Corollary.** If  $\{\varphi_n(x)\}$  is an orthonormal complete system on  $[0, 1]$ , then it can be rearranged so that the Lebesgue functions of the rearranged system  $\{\varphi_{m_i}(x)\}$  are unbounded at almost every point  $x \in [0, 1]$ .

We give one more result belonging to the same circle of questions:

**Theorem 9.** Let the orthonormal complete system  $\{\varphi_n(x)\}$  be uniformly bounded on  $[0, 1]$ . Then there exists a function  $F(x) \in L^p(0, 1)$  for all  $p > 0$  such that the series  $\sum a_n \varphi_n(x)$ , with

$$a_n = \int_0^1 F(t) \varphi_n(t) dt,$$

converges almost everywhere on  $[0, 1]$  and has property D.

This assertion follows directly from Theorem 6.

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*Note: Figure translations are in progress. See original paper for figures.*

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