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Abstract

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ON THE THEORY OF FRONTS

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On the basis of the solution of the equations of atmospheric hydrodynamics, developing the ideas of Margules (1), A. F. Dubyuk ((2) and others) and chiefly of F. K. Bol (3), we shall attempt to find the form of a frontal surface separating two thermally different air masses, and also to study the motion of the air that may occur near and directly on this surface.

For this purpose we simplify the system of equations of atmospheric hydrodynamics on the basis of the following assumptions: 1) the horizontal scales of the motion are of order 10^2–10^3 km (this allows us to use a Cartesian right-handed coordinate system (x, y, z), where x, y are directed horizontally, z upward, and to apply the equation of statics); 2) the perturbations of temperature and pressure are small in comparison with the mean values of these meteorological elements for the given height; 3) accelerations in the equations of motion may be neglected; 4) all elements of the motion do not depend on y, and consequently the frontal surface (on which a discontinuity of temperature takes place) is a cylindrical surface with generators directed along y; 5) the frontal surface moves, without changing its form, in the positive direction of the x-axis with a prescribed constant velocity c (always $c \geq 0$). Then in a coordinate system moving together with the front (we place the origin of coordinates on the Earth' s surface somewhere under the frontal surface), the process may be considered stationary.

After all these simplifications we arrive at the following initial system of equations:

$$-R\theta \frac{\partial}{\partial x}\left(\frac{p'_k}{P}\right) + l(v_k - v_g) + \nu \frac{\partial^2 u_k}{\partial z^2} = 0; \tag{1}$$

$$-l(u_k - u_g) + \nu \frac{\partial^2 v_k}{\partial z^2} = 0; \tag{2}$$

$$\frac{\partial u_k e^{-\sigma z}}{\partial x} + \frac{\partial w_k e^{-\sigma z}}{\partial z} = 0 \quad \left(\sigma = \frac{g - R\gamma}{R\theta}\right); \tag{3}$$

$$R\theta \frac{\partial}{\partial z}\left(\frac{p'_k}{P}\right) = \lambda \vartheta_k \quad \left(\lambda = \frac{g}{\theta}\right). \tag{4}$$

Here u_k, v_k, and w_k are the components of the air velocity (relative to the Earth) along the axes x, y, z, respectively; ϑ_k, p'_k are the deviations of the air temperature and pressure from the mean values of these meteorological elements θ and P for the given height (the air density has been eliminated from the system by means of the Clapeyron equation); u_g and v_g are the components of the geostrophic wind, which we shall regard as given and constant. The indices $k = 1$ and $k = 2$ refer to the cold and warm air masses, situated respectively below and above the frontal surface; g is the acceleration due to gravity; $l = 2\omega \sin\varphi$, where ω is the angular velocity of the Earth' s rotation and φ is the latitude of the place; ν is the kinematic coefficient of turbulent viscosity (the quantities l and ν are taken to be constant); R is the gas constant for air.

Let us make one more essential assumption, namely: suppose that the temperature perturbations are known, and specify them in the form $\vartheta_1 = \Delta\theta - z\Delta\gamma$, $\vartheta_2 = 0$, where $\Delta\theta$ is the difference between the temperatures at the Earth' s surface of the warm and cold air masses; $\Delta\gamma$ is the difference between the vertical temperature gradients of these masses. We shall regard the quantities $\Delta\theta$ and $\Delta\gamma$ as prescribed and constant.

The boundary conditions of the problem will be as follows:

$$u_1 = v_1 = w_1 = 0 \quad \text{for } z = 0; \tag{5}$$

$$u_1 = u_2, \quad v_1 = v_2, \quad \frac{\partial u_1}{\partial z} = \frac{\partial u_2}{\partial z}, \quad \frac{\partial v_1}{\partial z} = \frac{\partial v_2}{\partial z} \quad \text{for } z = h(x); \tag{6}$$

$$w_1 = w_2 = (u_1 - c)h'(x), \quad p'_1 = p'_2 \quad \text{for } z = h(x); \tag{7}$$

$$u_2 = u_g, \quad v_2 = v_g, \quad p'_2 = 0 \quad \text{for } z = \infty, \tag{8}$$

where $h(x)$ is the height of the frontal surface, to be determined in solving the problem.

Integrating equation (4) and satisfying conditions (7), (8) for p'_k, we find

$$\frac{p'_1}{P} = \frac{1}{R\theta}\left[\mu(h-z) + \frac{1}{2}m(h^2 - z^2)\right], \quad p'_2 \equiv 0, \tag{9}$$

where $\mu = -\lambda\Delta\theta > 0$, $m = \lambda\Delta\gamma$. Then equation (1) takes the form

Fig. 1

Figure 1: Fig. 1

$$l(v_k - v_g) + \nu \frac{\partial^2 u_k}{\partial z^2} = \mu h'(x) + mhh'(x) \tag{10}$$

(for $k = 2$ the right-hand side of this equation should be replaced by zero).*

Having integrated (3) with respect to z from 0 to h, taking (5), (7) into account, we obtain, instead of this equation, the integral condition

$$\int_0^h (u_1 - c)e^{-\sigma z}\, dz = \frac{Q}{\rho_0}, \tag{11}$$

where ρ_0 is the air density near the Earth' s surface; Q is the constant of integration, which from the physical point of view represents the quantity of air, in kilograms, penetrating in 1 sec. from the warm air mass into the cold one, referred to one running meter of the frontal surface. If the front rests on the Earth, then, obviously, $Q = 0$. If the front does not reach the Earth, as, for example, in the case of a warm front, which, as is known, often trails behind it a film of cold air, then determining Q is difficult. It is clear, however, that in comparison with the total amount of air moving in the frontal system, Q must be insignificant ([3]). Therefore we shall set $Q = 0$ always, thereby allowing in a number of cases a small error.

Fig. 1

Now one can indicate a way of solving the posed problem. First, from equations (2), (10), satisfying conditions (5), (6), (8), we find u_k and v_k

* It is not difficult to verify that for $m = \nu = 0$ equation (10) leads to the known Margules formula ([1]).

(the solution of this system is most simply carried out by constructing a single equation for $U = u_k + iv_k$).

In the present note we shall restrict ourselves to considering the case when one may set $\gamma_1 = \gamma_2$ ($m = 0$) and $\sigma = 0$. The latter simplification cannot introduce a large error into the calculations for the part of the front lying within the lower 3 km.

Since the derivations are elementary, we shall give only the final formulas:

$$\begin{aligned} u_1 &= u_2 + B(\eta - \xi), \quad u_2 = u + e^{-\xi}\,[A(\eta)\sin\xi - B(\eta)\cos\xi], \\ v_1 &= v_2 - A(\eta - \xi), \quad v_2 = v + e^{-\xi}\,[B(\eta)\sin\xi + A(\eta)\cos\xi]. \end{aligned} \tag{12}$$

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

Here the following notation is used: $\xi = z\sqrt{l}/\sqrt{2\nu}$; $\eta = h\sqrt{l}/\sqrt{2\nu}$; $A(\varphi) = (\operatorname{ch}\varphi\cos\varphi - 1) \times u_g\, d\varphi/d\xi$; $B(\varphi) = \operatorname{sh}\varphi\sin\varphi\, u_g\, d\varphi/d\xi$; $\xi = xlu_g\sqrt{l}/\mu\sqrt{2\nu}$; $u = u_g(1-e^{-\xi}\cos\xi) - v_g e^{-\xi}\sin\xi$; $v = u_g e^{-\xi}\sin\xi + v_g(1-e^{-\xi}\cos\xi)$. Having u_k, from (3), (5), and (7) we find ω_k by quadrature (for lack of space the formulas are not written out), after which, substituting u_1 into (12) and then integrating, we arrive at the equation for determining η:

$$\frac{d\eta}{d\xi} = -\frac{1 + V - 2(1-C)\eta - e^{-\eta}[(V+1)\cos\eta + (V-1)\sin\eta]}{{}^{3}/{}_{2} - 2e^{-\eta}(\cos\eta + \sin\eta) + {}^{1}/{}_{2}e^{-2\eta}(\cos 2\eta + \sin 2\eta)}, \tag{13}$$

where $V = v_g/u_g$; $C = c/u_g$.

Fig. 2

Numerical integration of (13) is elementary; however, it is necessary first to establish the character of its integral curves. Let us begin with an investigation of the behavior of the solution as $\eta \to 0$. Expanding the right-hand side in a series in powers of η, we obtain

$$\eta'(\xi) = \frac{1}{2}\left[-3C\eta^{-2} + 3(1-V)(2\eta)^{-1} + V + ...\right].$$

For sufficiently small η all terms except the first may be discarded. Then, integrating the resulting equation under the condition $\eta(0) = 0$, we shall have

$$\eta = (-4.5C\,\xi)^{1/3}.$$

In view of the fact that $C > 0$ and $\eta \geq 0$, it must be that $\xi \leq 0$. This means that, of the moving fronts, only a cold front can rest on the Earth, and moreover

Fig. 3

the section of the frontal surface immediately adjacent to the Earth must make a right angle with it*. It can be shown that the denominator of the fraction in (13) vanishes only for $\eta = 0$, while the numerator may have one or two zeros. Consequently, the integral curves may have one or two asymptotes (or else have none at all). The values $\eta = \eta_a$ and $\eta = \eta_b$ ($\eta_b > \eta_a$), at which the frontal surfaces become horizontal, are easily found graphically for given V and C with the aid of the curves $C = \text{const}$ in Fig. 1.

Fig. 4

Figure 4: Fig. 4

Further analysis shows that, depending on the presence or absence of asymptotes, the integral curves of equation (13) may be of 5 types. The schematically possible types of integral curves** are shown in Fig. 2. Curves 1 and 2 correspond to cold fronts, curves 3 and 4 to warm ones. Branch of the solution 5 has no physical meaning, since in this case the condition $Q = 0$ ceases to be valid. As is easily seen from Fig. 2, a cold air mass may be bounded in height (curves 2 and 4) or unbounded (curves 1 and 3). Curve 3 shows that a warm front must always pull along behind it a film of cold air, the thickness of which may range from several tens of meters to several hundreds of meters. This conclusion agrees with the results of observations in nature.

Fig. 4

After equation (13) has been solved and $h(x)$ found (with Fig. 1 making it possible to choose the required branch of the solution), the calculation of the velocity and pressure fields in the air masses and in the frontal region is carried out by formulas (9), (12), etc. The results of calculations of examples are given in Figs. 3 and 4. In Fig. 3 a section through a warm front is plotted ($\Delta\theta = -10^\circ$; $C = 0.25$; $V = -0.5$) in a coordinate system moving together with the front. The projection of the frontal surface onto the vertical plane perpendicular to the front line is indicated by a heavy line; the corresponding projections of the stream surfaces, by thin lines. The arrows indicate the direction of motion of the air. The dashed lines show the height distributions of the relative velocity components v_k/u_g, corresponding to two values of ξ. Fig. 4 gives the case of a cold front ($\Delta\theta = -10^\circ$; $C = 1.5$; $V = 1.5$).

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* Corresponding conclusions can also be obtained for a stationary front, when

$C = 0$.

** In their character the integral curves coincide with the curves obtained by Ball (3), although the latter uses a cruder hydraulic formulation of the problem.

Note: Figure translations are in progress. See original paper for figures.