

# ON CONDITIONS FOR THE EXISTENCE OF A SOLUTION OF A QUASILINEAR PARABOLIC EQUATION

Existence theorems for the solution of the first boundary-value problem for the parabolic equation

1961

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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON CONDITIONS FOR THE EXISTENCE  
OF A SOLUTION OF A QUASILINEAR  
PARABOLIC EQUATION**

*(Presented by Academician I. G. Petrovskii on 3 VII 1961)*

Existence theorems for the solution of the first boundary-value problem for the parabolic equation

$$u_t = a(t, x, u, u_x)u_{xx} + b(t, x, u, u_x), \quad (1)$$

$$u(0, x) = \varphi_0(x), \quad u(t, X_1) = 0, \quad u(t, X_2) = 0 \quad (2)$$

in the domain  $0 \leq t \leq T$ ,  $X_1 \leq x \leq X_2$  ( $T$  arbitrary) were established in works <sup>(1-4)</sup>. In doing so, restrictions of three types were imposed on the functions  $a$  and  $b$ : 1) restrictions on the growth rate of the ratio  $b/a$  as  $|u_x| \rightarrow \infty$ ; 2) smoothness of the functions  $a$  and  $b$ ; 3) the requirement that certain derivatives of the functions  $a$  and  $b$  (or certain combinations of them) have a definite sign. In the present note an existence theorem for a solution is established in which no restrictions of type 3) are imposed, and the restrictions of type 1) are weaker than in works <sup>(1-4)</sup>. In the case when the ratio  $b/a$  does not depend on  $t$  and  $x$ , the imposed restriction on the growth rate admits no further weakening, since for equations of the form  $u_t = u_{xx} + uf(u_x)$  it is not only sufficient but also necessary for the existence of a solution under arbitrary boundary conditions.

**Theorem 1.** Let, in equation (1), the functions  $a$  and  $b$  be three times continuously differentiable,  $a(t, x, u, u_x) \geq a_0 > 0$ ,

$$ub(t, x, u, 0) \leq |u|\psi(|u|), \quad \psi \geq 1, \quad \int_0^\infty \frac{du}{\psi(u)} = \infty, \quad (3)$$

and for any  $U$ , when  $|u| \leq U$ ,  $|v| \geq V_0$ , let

$$|q(t, x, u, v)| \leq Cv^2, \quad |q_v| \leq C_1|v|, \quad |q_t| \leq C_2v^2, \quad |q_x| \leq \alpha(v)|v|^3, \quad (4)$$

where  $q = b/a$ ;  $\alpha(v) \rightarrow 0$  as  $|v| \rightarrow \infty$ ;  $C, C_1, C_2, \alpha, V_0$  may depend on  $U$ . Let the function  $\varphi_0$  in (2) satisfy the Hölder condition and the compatibility conditions

$$\varphi_0(X_i) = 0, \quad a(0, X_i, \varphi_0(X_i), \varphi_0'(X_i))\varphi_0''(X_i) + b(0, X_i, \varphi_0(X_i), \varphi_0'(X_i)) = 0,$$

$$i = 1, 2.$$

Then the boundary-value problem (1), (2) in the domain  $D: (0 \leq t \leq T, X_1 \leq x \leq X_2)$ , where  $T$  is arbitrary, has a unique solution in the class of functions continuous in  $D$  and possessing the derivatives  $u_t, u_x, u_{xx}$  inside  $D$ . For this solution  $u_t, u_x, u_{xx}$  are continuous in  $D$  and satisfy the Hölder condition in  $t$  and  $x$ ; inside  $D$  there exist continuous  $u_{tt}, u_{txx}, u_{xxxx}$ .

**Remark.** The theorem remains valid if condition (4) is replaced by any of the following two conditions:

$$\text{A.} \quad \left| \frac{\partial}{\partial v} \left( \frac{q}{v^2} \right) \right| \leq \frac{C_1}{|v|}, \quad |q_t| \leq C_2 v^2 \ln |v|, \quad |q_x| \leq C_3 v^2 (\ln |v|)^{c_2};$$

or

B.  $q = q(u, v)$  does not depend on  $t$  and  $x$ ,  $|q| \leq |v|\psi(|v|)$ , where the function  $\psi$  is the same as in (3).

**Theorem 2.** For the equation  $u_t = u_{xx} + uu_x\psi(u_x)$ , where the function  $\psi$  has continuous  $\psi'''$  and  $|\psi(v)| \geq 1, v\psi(v) > 0$  for  $|v| \geq V_0$ , the divergence of the integral

$$\int_{\pm V_0}^{\pm \infty} \frac{dv}{\psi(v)}$$

is a necessary and sufficient condition in order that, for arbitrary smooth initial and boundary conditions and any  $T$ , the solution of the first boundary-value problem exist for  $0 \leq t \leq T$ .

**Theorem 3.** Let the functions  $u(t, x), v(t, x)$  be continuous in the bounded closed domain  $D_1$ , lying in the strip  $0 \leq t \leq T$ , and let  $u_x, v_x$  and the left derivatives  $u_t^-, v_t^-, u_{xx}^-, v_{xx}^-$  exist everywhere inside  $D_1$ . Let

$$Lu \equiv u_t^- - f(t, x, u, u_x, u_{xx}^-),$$

where the function  $f$  is nondecreasing in  $u_{xx}^-$ ;  $f$  and  $\partial f / \partial u$  are continuous. If: a) on the set  $\Gamma(\bar{D}_1)$  of those boundary points of the domain  $D_1$  at which  $t < T$ , we have  $u \geq v$ ; b) inside  $D_1, Lu \geq Lv$ ; c)  $u_x, u_{xx}^-$  (or  $v_x, v_{xx}^-$ ) are bounded in  $D_1$ , then  $u \geq v$  throughout the domain  $D_1$ .

**Remark.** The following example shows that condition c) cannot be dropped. Let

$$Lu \equiv u_t - 9u_{xx} - 2uu_x^3 - bu_x^4(u^3 - 27x),$$

$D_1$  be the domain  $0 \leq t \leq 1$ ,  $0 \leq x \leq 1$ ;  $u = 3x^{1/3}$ ,  $v = u + x^3(1-x)\xi e^{-\xi^2}$ ,  $\xi = xt^{-1/2}$ . Then on  $\Gamma(\bar{D}_1)$ ,  $u = v$ , while inside  $D_1$ ,  $Lu = 0$ ,  $Lv < 0$ , if the number  $b$  is sufficiently large. However  $u < v$  inside  $D_1$ .

**Proof of Theorem 1.** The uniqueness of the solution follows from Theorem 3. The existence of a solution on a sufficiently small time interval  $0 \leq t \leq T_1$  follows from (4-7). This solution, in the closed domain  $0 \leq t \leq T_1$ ,  $X_1 \leq x \leq X_2$ , has derivatives  $u_t, u_x, u_{xx}$  satisfying the Hölder condition in  $t$  and  $x$ . Denote by  $T_2$  the upper bound of such values of  $T_1$ . Then in the domain  $D_2$  ( $0 \leq t < T_2$ ,  $X_1 \leq x \leq X_2$ ) the solution exists and is continuous together with the derivatives  $u_t, u_x, u_{xx}$ , while inside  $D_2$  there exist continuous  $u_{tt}, u_{txx}, u_{xxxx}$  (6).

We shall show that  $u, u_x, u_{xx}, u_t$  are bounded in  $D_2$ . The boundedness of  $u$  is proved by comparison, by Theorem 3, of the solution  $u(t, x)$  of problem (1), (2) with the solution of the ordinary differential equation  $dz/dt = \psi(z)$ . Thus,  $|u| \leq M$  in the domain  $D_2$ .

We shall prove the estimate of  $u_x$  on  $\Gamma(D_2)$ , i.e., on that part of the boundary of the domain  $D_2$  where  $t < T_2$ . Put

$$U(t, x) = y(x - X_1 - d) + M,$$

where  $y(x)$  is the solution of the equation  $y'' + y'\psi(y') = 0$  with initial conditions  $y(0) = 0$ ,  $y'(0) = V_1$ ; the function  $\psi$  is such that  $|q| \leq |v|\psi(|v|)$  for  $|v| \geq V_0$ ,  $\psi \geq 1$ ; the integral (3) diverges,  $V_1 = \max\{V_0; \varphi'_0\} + 1$ , and the number  $d > 0$  is such that  $y(-d) = -M$ . Then for  $t = 0$ , for  $x = X_1$ , and for  $x = X_3 = \min\{X_1 + d; X_2\}$  we have  $u(t, x) \leq U(t, x)$ , while for  $X_1 < x < X_3$  we have  $0 \leq U \leq M$ ,  $V_1 \leq U_x \leq y'(-d)$ ,  $LU > 0$ ,  $Lu = 0$ . By Theorem 3 we have  $u \leq U$ . Since  $u(t, X_1) = U(t, X_1) = 0$ , it follows that  $u_x(t, X_1) \leq U_x(t, X_1) = y'(-d)$ . Similarly,  $-u_x(t, X_1) \leq y'(-d)$ .

For estimating  $u_x$  inside the domain  $D_2$ , note that if  $\varphi(t, x, u, v) = C$ —the first integral of the system

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{v} = -\frac{dv}{q(t, x, u, v)}, \quad (5)$$

with  $\varphi_v > 0$ , then equation (1) is written in the form

$$u_t = m(t, x, u, u_x) \frac{d}{dx} \varphi(t, x, u, u_x),$$

where  $m = a/\varphi_v$ .

Taking as the new unknown function  $p(t, x) = \varphi(t, x,$

$u(t, x), u_x(t, x)$ , we obtain an equation of the form  $p_t = ap_{xx} + kp_x + \varphi_t$ , where  $k$  is a continuous function. Now, in order to prove boundedness in  $D_2$  of the function  $p$ , and then of the function  $v = u_x$ , it is enough to require that  $v$  be bounded on  $\Gamma(D_2)$  (which has already been proved), and that the function  $\varphi$  in the region  $0 \leq t < T_2$ ,  $X_1 \leq x \leq X_2$ ,  $|u| \leq M$ ,  $|v| \geq V^*$  ( $V^*$  is some number) have the following properties: a)  $\varphi$  is three times continuously differentiable; b)  $\varphi_v > 0$ ; c)  $|\varphi_t| \leq \psi(|\varphi|)$  and the integral (3) diverges; d) as  $|v| \rightarrow \infty$  we have  $|\varphi| \rightarrow \infty$ , and conversely.

We obtain such a function  $\varphi$  if, for every sufficiently large  $|C|$ , we set  $\varphi(t, x, u, v) = C$  at the points of all integral curves of the system (5) satisfying the condition  $v = C$  when  $u = -M$  (in (5) we regard  $u$  as an independent variable,  $-M \leq u \leq M$ ). From the inequality  $|q| \leq |v|\psi(|v|)$  it follows that  $\varphi$  has property d). To prove properties b) and c), we note that  $\varphi_v$  and  $\varphi_t$  may be regarded as derivatives of the solution of system (5) with respect to the initial conditions; therefore their estimate reduces to the estimate of the solution of some auxiliary system of linear differential equations. We obtain that, when condition (4) (or conditions A or B) is fulfilled, the function  $\varphi$  has properties b) and c).

After the boundedness of  $u$  and  $u_x$  has been proved, it is not difficult to prove the boundedness of  $u_{xx}$  and  $u_t$ . From the boundedness of the latter it follows that  $u_x$  satisfies a Hölder condition in  $t$  and  $x$  throughout the whole region  $D_2$  (briefly:  $u_x \in H$ ). Hence,  $a(t, x, u(t, x), u_x(t, x)) = A(t, x) \in H$ ,  $b(t, x, u(t, x), u_x(t, x)) = B(t, x) \in H$ . Since  $u_t = A(t, x)u_{xx} + B(t, x)$ , by virtue of (5),  $u_t \in H$ ,  $u_{xx} \in H$ . Hence, if at  $t = T_2$  the function  $u$  is extended by continuity, then  $u(T_2, x)$  will satisfy a Hölder condition in  $x$ ; by virtue of the continuity of  $u_t, u_x, u_{xx}$ , the compatibility conditions at the points  $(T_2, X_1)$  and  $(T_2, X_2)$  will be fulfilled. If  $T_2 < T$ , then with such initial conditions  $u = u(T_2, x)$ , prescribed at  $t = T_2$ , a solution  $u(t, x)$  for which  $u_t \in H$ ,  $u_{xx} \in H$  exists also for  $T_2 \leq t \leq T_2 + \tau$ ,  $\tau > 0$ . This contradicts the choice of the number  $T_2$ . Hence  $T_2 = T$ . Theorem 1 is proved.

In Theorem 2, sufficiency follows from Theorem 1. To prove necessity, suppose that the integral referred to in the theorem converges. Denote by  $y(x)$  the solution of the equation  $y'' + y'\psi(y') = 0$  with initial conditions  $y(0) = 0$ ,  $y'(0) = +\infty$ . Then, for sufficiently small  $h$ , we have  $0 < y < 1$ ,  $V_0 < y' < \infty$ ,  $y'' < 0$  for  $0 < x \leq h$ ;  $y'(h) > 2$ . Let  $m(t)$  be such a function that  $y'(m(t)) = \frac{N}{1-t}$ ,  $N = y'(h) > 2$ ,  $0 < m(t) \leq h$  for  $0 \leq t < 1$ . Put

$$v(t, x) = \begin{cases} \frac{Nx}{1-t}, & \text{for } 0 \leq x \leq c(t) = \frac{y(m(t)) + N}{y'(m(t))}, \\ y(x - c(t) + m(t)) + N, & \text{for } c(t) \leq x \leq h. \end{cases}$$

Obviously, for  $0 \leq x \leq h$ ,  $0 \leq t < 1$ , the function  $v$  is continuously differentiable.

It can be shown that  $v_t - v_{xx} - vv_x \psi(v_x) \leq 0$ . Suppose that there exists a solution  $u(t, x)$ , continuous for  $0 \leq x \leq h$ ,  $0 \leq t \leq 1$ , of the equation of Theorem 2, for which  $u(t, 0) = 0$ , and, at  $t = 0$  and at  $x = h$ ,  $u(t, x) \geq v(t, x)$ . Then, by Theorem 3, for arbitrarily small  $\delta > 0$  and  $0 \leq t \leq 1 - \delta$ , we must have  $u \geq v$ . This contradicts the continuity of  $u(t, x)$ , since  $u(1, 0) = 0$ , whereas  $v(t, 1 - t) = N > 2$ .

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Received  
29 VI 1961

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*Note: Figure translations are in progress. See original paper for figures.*

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