MATHEMATICS

1961

SovietRxiv

Abstract

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MATHEMATICS

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CAUCHY DIRECTIONS AND EXTENSIONS OF REGULAR SPACES

(Presented by Academician P. S. Aleksandrov on 14 X 1960)

It is known that the Cantor-Hausdorff process of completion of metric spaces has been generalized to uniform spaces (^{1}). In the present paper it is shown that this process is essentially not connected with the concept of a uniform structure and can be generalized at least to all possible regular extensions of regular spaces. This is achieved by introducing the notion of a Cauchy direction, which generalizes the classical notion of a Cauchy sequence.

1. In what follows we consider Hausdorff topological spaces. Denote by $\mathfrak{U}$ such an open base of the space R which has the following properties: 1) if U and $V \in \mathfrak{U}$, then $U \cup V$ and $U \cap V \in \mathfrak{U}$; 2) if $U \in \mathfrak{U}$, then $R - [U_R] \in \mathfrak{U}$, where $[H]_R$ denotes the closure of the set H in the space R. The sets of the base $\mathfrak{U}$ are called **neighborhoods** of the space R.

A nonempty system I of sets of the base $\mathfrak{U}$ is called its α-**ideal** if the following conditions are satisfied: a) from $U \in I$ it follows that $U \neq \Lambda$ (Λ denotes the empty set); b) from U and $V \in I$ it follows that $U \cap V \in I$; c) from $U \in I$, $V \in \mathfrak{U}$ and $V \supset U$ it follows that $V \in I$. An α-ideal I is called **primary** if: d) from $U = V \cup W$, where $U \in I$, and V and $W \in \mathfrak{U}$, it follows that at least one of the sets V and W belongs to I (^{2}).

Let $\Gamma(\prec)$ denote a directed partially ordered set (p.o.s.) with transitive relation $\prec$ (see (^{3})). An end of a direction Γ is such a subset of it which consists of all elements $\gamma \prec \gamma_0$ for some $\gamma_0 \in \Gamma$. The weight of a p.o.s. Γ is the least cardinal τ for which there exists a subset $\Gamma_1 \subseteq \Gamma$ of cardinality τ, cofinal ($\widetilde{\subset}$)Γ.

A direction of sets in R is any (nonempty) system $\{H_\gamma\}_\Gamma$ of its subsets, indexed by means of a directed p.o.s. Γ. If $\Gamma_1 \widetilde{\subset} \Gamma$, then $\{H_\gamma\}_{\Gamma_1}$ is called a subdirection of the direction $\{H_\gamma\}_\Gamma$. Every α-ideal I of the base $\mathfrak{U}$ can be regarded as a direction of sets by inclusion. $I = \{U_\gamma\}_\Gamma$, where $\gamma_1 \prec \gamma_2$ ($\gamma_1, \gamma_2 \in \Gamma$) means that $U_{\gamma_1} \subseteq U_{\gamma_2}$.

The least cardinality greater than or equal to the weight of every primary α-ideal of the base $\mathfrak{U}$ is called the **local weight of the space** R **relative to the base** $\mathfrak{U}$ and is denoted by $\tau(R, \mathfrak{U})$.

Let $x \in R$. In what follows I_x denotes the system of all neighborhoods of the point x. I_x is a primary α-ideal of the base $\mathfrak{U}$. The weight $\tau(x)$ of the direction of sets I_x does not depend on the choice of the base $\mathfrak{U}$ and is called the **weight of the space R at the point x**. The least cardinality greater than or equal to $\tau(x)$ for every $x \in R$ is called the **local weight of the space R** and is denoted by $\tau(R)$. Obviously,

$$\tau(R, \mathfrak{U}) \geq \tau(R).$$

The **upper topological limit** $\overline{\mathrm{lt}}_\Gamma H_\gamma$ of the direction of sets $\{H_\gamma\}_\Gamma$ is the set $\overline{H}$ of all points x such that for every neighborhood $U(x)$ of them there exists a $\Gamma_1 \subset \Gamma$ such that $U(x) \cap H_\gamma \neq \Lambda$ for $\gamma \in \Gamma_1$. The **lower topological limit** $\underline{\mathrm{lt}}_\Gamma H_\gamma$ of the direction of sets $\{H_\gamma\}_\Gamma$ is the set H of all points x every neighborhood $U(x)$ of which intersects almost all H_γ (i.e. $U(x) \cap H_\gamma \neq \Lambda$ for $\gamma \preccurlyeq \gamma_0$) (see (4^5)). The direction $\{H_\gamma\}_\Gamma$ will be called **convergent** if

$$\overline{\mathrm{lt}}_\Gamma H_\gamma = \underline{\mathrm{lt}}_\Gamma H_\gamma = H \neq \Lambda,$$

and **conditionally convergent** if

$$\overline{\mathrm{lt}}_\Gamma H_\gamma = \underline{\mathrm{lt}}_\Gamma H_\gamma = \Lambda.$$

2. In the case when the H_γ are singleton sets, we speak of a direction of points $X = \{x_\gamma\}_\Gamma$. Every direction of points can converge only to a point. In this case we write

   $$\mathrm{lt}_\Gamma\, x_\gamma = x,$$

   or

   $$x_\gamma \xrightarrow{\Gamma} x.$$

   In order that a set $H \subset R$ be closed, it is necessary and sufficient that it contain the limits of all its convergent directions of points of cardinality $\leq \tau(R)$. In order that a mapping $y = f(x)$ of the space R into the space R' be continuous, it is necessary and sufficient that from

   $$x_\gamma \xrightarrow{\Gamma} x$$

   in R it follow that

   $$f(x_\gamma) \xrightarrow{\Gamma} f(x)$$

   in R' for every convergent direction of points $\{x_\gamma\}_\Gamma$ of cardinality $\leq \tau(R)$ (cf. (6)).

A direction of points $X = \{x_\alpha\}_A$ is called a **Cauchy direction**, or, briefly, a **K-direction**, with respect to a **base** $\mathfrak{U}^*$ if there exists a primary α-ideal

$$I_X = \{U_\gamma\}_\Gamma$$

of the base $\mathfrak{U}$, possessing the property:

(1) For every $\gamma \in \Gamma$ there exists an index $\alpha(\gamma) \in A$ such that for $\alpha \preccurlyeq \alpha(\gamma)$ one has $x_\alpha \in U_\gamma$.

Every subdirection Y of a K-direction X is itself a K-direction, and as I_Y one may take any I_X. Every convergent direction of points of the space R is a K-direction.

If X and Y are any two K-directions converging to one and the same point, then there exists such a primary α-ideal I of the base $\mathfrak{U}$ that

$$I = I_X = I_Y.$$

For every primary α-ideal I there exists a K-direction X of cardinality $\leq \tau(R, \mathfrak{U})$** for which $I = I_X$.

If all neighborhoods of the α-ideal I of the base $\mathfrak{U}$ have nonempty intersection with some set $H \subseteq R$, then there exists a K-direction X of cardinality

$$\leq \tau(R, \mathfrak{U}),$$

consisting of points of the set H, for which $I_X \supset I$.

In a regular space every K-direction either converges or conditionally converges.

In order that a K-direction $X = \{x_\alpha\}_A$ converge in a regular space, it is necessary and sufficient that

$$\overline{\mathrm{lt}}_A x_\alpha \neq \Lambda.$$

If in a regular space a K-direction X converges to a point x, then

$$I_X \supset I_x.$$

For every K-direction X of a regular space R, I_X contains every neighborhood U of the base $\mathfrak{U}$ for which there exists a neighborhood $V \in \mathfrak{U}$ such that

$$U \supset [V]_R,$$

and V contains some tail of the direction X.

3. As usual, a space R^+ is called an **extension of the space** R if R is homeomorphic to some subspace R' of the space R^+, everywhere dense in R^+.

Theorem 1. Let R be a regular space; $\mathfrak{U}$ some open base of it; $\mathfrak{K}$ the set of all K-directions in R with respect-

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* We shall simply speak of a K-direction where this causes no misunderstanding.

** By the cardinality of the direction $\{x_\gamma\}_\Gamma$ is meant the cardinality of the set Γ.

a base $\mathfrak{U}$ of cardinality $\leq \tau(R, \mathfrak{U})$; $\mathfrak{R}^+$ is a subset of $\mathfrak{R}$, taken together with some equivalence relation $\sim$ such that:

I. Every $X \in \mathfrak{R}$ converging in R belongs to $\mathfrak{R}^+$.

II. If

$$z = \{X_\delta / X_\delta \in \mathfrak{R}^+\}_{\delta \in \Delta}$$

is a certain class of mutually equivalent K-directions, and $\{I^l_{X_\delta}\}_{l \in L(\delta)}$ is the system of all corresponding X_δ, according to the definition in item 2, of prime a-ideals of the base $\mathfrak{U}$, and

$$I_z = \bigcap_{\delta \in \Delta} \bigcap_{l \in L(\delta)} I^l_{X_\delta},$$

then the class z contains every K-direction $X \in \mathfrak{R}$ for which there exists $I_X \supset I_z$.

III. If X and $Y \in \mathfrak{R}^+$, and: a) if $X \to x$ and $Y \to x$ in R, then $X \to Y$; b) if $X \to x$ and $Y \to y$ in R and $x \neq y$, then $X \nsim Y$; c) if $X \to x$, while Y converges conditionally in R, then $X \nsim Y$.

Denote by R^+ the set of all classes of mutually equivalent K-directions from $\mathfrak{R}^+$. Define convergence in R^+ by the rule:

IV. A direction of points $\{z_\alpha\}_A$ of the set R^+ will be called convergent to the point $z \in R^+$ if for every neighborhood $U \in J_z$ there is an index $\alpha_0 \in A$ such that, for $\alpha > \alpha_0$, U contains some tail of any K-direction $X_\alpha \in z_\alpha$.

Select in R^+ the system of sets $\mathfrak{F}^+$ containing the limits of all their convergent directions of points of cardinality $\leq \tau(R, \mathfrak{U})$. We shall take it as the system of closed sets in R^+. Then R^+ will be a T_1-space that is an extension of the space R. In this case the function $x' = \varphi(x)$, assigning to each point $x \in R$ the class x' containing the stationary K-direction generated by the point x, realizes a homeomorphic mapping of the space R onto an everywhere dense subspace R' of R^+.

Theorem 2. Let R be a regular space; let $\hat{R}^+$ be some regular extension of it; let $\hat{R}'$ be a subspace of $\hat{R}^+$, dense in $\hat{R}^+$ and homeomorphic to R under the mapping $\hat{x} = \hat{\varphi}(x)$; let $\hat{\mathfrak{U}}^+$ be some open base of the space $\hat{R}^+$; $\hat{\mathfrak{U}}' = \hat{\mathfrak{U}}^+ \cap \hat{R}'$; $\mathfrak{U} = \hat{\varphi}^{-1}(\hat{\mathfrak{U}}')$.

Denote further by $\mathfrak{R}$ the set of all K-directions in R relative to the base $\mathfrak{U}$ of cardinality $\leq \tau(R, \mathfrak{U})$; by $\mathfrak{R}^+$, the subset of those K-directions from $\mathfrak{R}$ whose images under the mapping $\hat{\varphi}$ converge in $\hat{R}^+$. Introduce in $\mathfrak{R}^+$ an equivalence relation, putting $X \sim Y$ $(X, Y \in \mathfrak{R}^+)$ if $\hat{X} = \hat{\varphi}(X)$ and $\hat{Y} = \hat{\varphi}(Y)$ converge in $\hat{R}^+$ to one and the same point. Then this relation will satisfy all the conditions of Theorem 1, and the space R^+ generated by it, according to that theorem, will be homeomorphic to $\hat{R}^+$. Under this homeomorphism the spaces R' and $\hat{R}'$ are mapped onto one another by the function

$$\hat{x} = \hat{\varphi}[\varphi^{-1}(x)].$$

4. Let R denote a metric space; $\mathfrak{U}$, its open base generated by all open balls. The connection between the notions of a Cauchy direction relative to the

base $\mathfrak{U}$ in the sense of item 2 and of a Cauchy sequence in the classical sense is established by the following propositions.

It is known that every Cauchy sequence in the classical sense of the word is a totally bounded set. It turns out that every totally bounded Cauchy sequence relative to the base $\mathfrak{U}$ in the sense of item 2 is also a Cauchy sequence in the classical sense of the word. The converse proposition, generally speaking, is false. However, in the case when the local weight $\tau(R, \mathfrak{U})$ is countable, from every Cauchy sequence in the classical sense of the word one can extract a subsequence which is a (totally bounded) Cauchy sequence in the sense of item 2.

The question of whether there exist metric spaces whose local weight relative to the base $\mathfrak{U}$ is uncountable remains open.

5. We shall call the space R **complete relative to its open base** $\mathfrak{U}$, if every K-direction in R relative to

of this base converges in R. We shall call the space R **absolutely complete** if it is complete with respect to each of its open bases.

It turns out that every bicompact space is an absolutely complete space. In order that a normal space R be bicompact, it is sufficient that it be complete with respect to the system of all open sets of this space.

From these two propositions one obtains the following characterization of bicompact spaces.

Theorem 3. *In order that a normal space R be bicompact, it is necessary and sufficient that one of the following two conditions hold:*

I. *R is complete with respect to the system of all its open sets.*

II. *R is absolutely complete.*

Received
29 III 1960

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Note: Figure translations are in progress. See original paper for figures.