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SEMIGROUPS WITH CERTAIN TYPES OF SUBSEMIGROUP STRUCTURES

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The present note is a continuation of our note ([1]), where, in particular, the basic definitions were given and some general considerations on the structural properties of semigroups were stated. Here we shall consider semigroups whose subsemigroup structures possess one or another of the following properties: linearly ordered, distributive, with complements. We note that Theorems 1, 2, and 4 at the same time give a structural characterization of the resulting classes of semigroups (Theorem 4—in the class of commutative semigroups).

Let us recall that by $\Sigma(\Gamma)$ we denote the partially ordered by inclusion set of all subsemigroups of the semigroup Γ, and by $\Sigma'(\Gamma)$ the extended set obtained by adjoining the empty set to $\Sigma(\Gamma)$. In addition to the remarks concerning $\Sigma(\Gamma)$ and $\Sigma'(\Gamma)$ made in ([1]), we note that it is always necessary to indicate whether we are dealing with $\Sigma(\Gamma)$ or with $\Sigma'(\Gamma)$, which is especially important in those cases when the properties of $\Sigma(\Gamma)$ and $\Sigma'(\Gamma)$ differ substantially. For example, if $\Sigma(\Gamma)$ is a structure with complements, then $\Sigma'(\Gamma)$ is not a structure with complements; on the other hand, if $\Sigma'(\Gamma)$ is a structure with complements, then $\Sigma(\Gamma)$ is not a structure at all (all this except for the trivial case when Γ is one-element). Imposing these conditions on $\Sigma(\Gamma)$ and $\Sigma'(\Gamma)$ leads to different classes of semigroups (see Theorems 3 and 4).

Let $\{x\}$ be the finite cyclic subsemigroup generated by the element x, and let m be the least natural number with the property that $x^h = x^m$, $h < m$. Put $d = m - h$. The pair of numbers (h, d) is called the **type** of the semigroup $\{x\}$, and also the type of the element x (see ([2])).

Theorem 1. *In order that $\Sigma(\Gamma)$ be a linearly ordered set, it is necessary and sufficient that it be either a finite ordered set, or an ordered set of type $\omega + 1$. For the first case it is necessary and sufficient that Γ be a cyclic semigroup of type (h, d), where $1 \leq h \leq 3$, $d = p^n$, p is prime, $n \geq 0$, and, if $h = 3$, then $p \neq 2$ ([3]). For the second case it is necessary and sufficient that Γ be a group of type p^∞.*

Remark at proof correction. An analogous result is contained in the work ([9]), of which the author learned from *Referativnyi Zhurnal* only after the present article had been sent to press.

We proceed to the description of semigroups with a distributive subsemigroup structure. This description is given by Theorem 2, whose proof rests on a number of lemmas stated below.

Lemma 1. *If $\Sigma'(\Gamma)$ is a Dedekind (in particular, distributive) structure, then Γ is a periodic semigroup in which every element has type (h, d), where $h \leq 5$.*

Let Γ be an arbitrary periodic semigroup. As is known, then each of its subsemigroups contains an idempotent. By K_e we shall denote the set of all elements of Γ some power of which is equal to the idempotent e. Γ decomposes into the set-theoretic sum of pairwise disjoint classes K_e. In the general case each class K_e need not be a subsemigroup in Γ. However, in our case we have:

Lemma 2. *If $\Sigma'(\Gamma)$ is distributive, then each class K_e of the semigroup Γ is its subsemigroup.*

A semigroup Γ is called a **bundle** ([2,4]) of its subsemigroups $\Gamma_\alpha, \Gamma_\beta, \ldots$, called the **components** of the bundle, if all components are pairwise disjoint, their set-theoretic sum is equal to Γ, and for every pair Γ_ξ, Γ_η of these subsemigroups there is a subsemigroup Γ_σ such that $\Gamma_\xi \Gamma_\eta \subseteq \Gamma_\sigma$. We shall call a bundle **strong** if, for arbitrary subsemigroups H_α and H_β from different components, their composite coincides with the set-theoretic sum:

$$\{H_\alpha, H_\beta\} = H_\alpha \cup H_\beta.$$

Lemma 3. *In order that $\Sigma'(\Gamma)$ be distributive, it is necessary and sufficient that Γ be a strong bundle of its subsemigroups Γ_α, each of which contains one idempotent, and $\Sigma'(\Gamma_\alpha)$ be distributive for all Γ_α.*

Lemma 3 reduces the study of arbitrary semigroups with a distributive structure of subsemigroups to the study of such semigroups containing one idempotent.

Let D be an ideal* of an arbitrary semigroup Γ. By $\Gamma - D$ we denote the quotient semigroup of Γ by D (in the sense of Rees ([5])). We say that Γ is an **extension** of the semigroup A by means of the semigroup B, if Γ has an ideal isomorphic to A, and the quotient semigroup by it is isomorphic to B ([6]). Every periodic semigroup Γ with one idempotent is an extension of its kernel G (i.e. the least ideal that is a group) by means of a nilsemigroup ([1]). If $\Sigma'(\Gamma)$ is distributive, then, by Ore' s theorem ([7]), G is a periodic locally cyclic group, whose structure, as is known, admits a complete description. Let us consider the quotient semigroup $\Gamma - G$.

Lemma 4. *Let $\Sigma'(\Gamma)$ be distributive, Γ contain a unique idempotent, D be an ideal in Γ, φ be the ideal homomorphism of Γ onto $\Gamma - D$. Then the one-to-one mapping of the structure $\Sigma'(\Gamma)$ onto the structure $\Sigma'(\Gamma - D)$, induced by the homomorphism φ, is a homomorphism of $\Sigma'(\Gamma)$ onto $\Sigma'(\Gamma - D)$, and therefore $\Sigma'(\Gamma - D)$ is also distributive.*

It follows from Lemma 4 that $\Gamma - G$ is a nilsemigroup with a distributive structure of subsemigroups. In ([8]) we described this class of semigroups. We shall state the corresponding result in the form of a lemma.

Lemma 5. *For a nilsemigroup* Γ, $\Sigma(\Gamma)$ *will be a distributive structure if and only if the composite of any two of its subsemigroups coincides with their set-theoretic sum. It follows from this that* Γ *is nilpotent of class* ≤ 5 ($\Gamma^5 = 0$, *where* 0 *is the zero of* Γ).

By $Z(\Gamma)$ we denote the center of the semigroup Γ, i.e. the set of all elements of Γ that commute with every element of Γ.

Lemma 6. *Let* D *be an ideal of an arbitrary semigroup* Γ. *Then*

$$Z(D)^2 \subseteq Z(\Gamma).$$

Corollary. *If a commutative ideal of a semigroup coincides with its square, then it is contained in the center of the semigroup.*

The corollary from Lemma 6, Lemmas 4 and 5 are used in the proof of the following lemma, describing the structure of a semigroup with one idempotent possessing a distributive structure of subsemigroups.

Lemma 7. *If* Γ *is a semigroup with one idempotent, then* $\Sigma'(\Gamma)$ *is distributive if and only if* Γ *contains a kernel* G, *which is a periodic locally cyclic group, and the quotient semigroup* $\Gamma - G$ *is a nilpotent semigroup in which the composite of any two subsemigroups coincides with their set-theoretic sum.*

From Lemmas 3 and 7 the following follows:

Theorem 2. $\Sigma'(\Gamma)$ *will be distributive if and only if the semigroup* Γ *is a strong bundle of its subsemigroups* Γ_α, *each of which possesses a kernel* G_α, *which is a periodic locally cyclic–*

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* By an ideal here and throughout is meant a two-sided ideal.

group, and the factor semigroup Γ_α / G_α is a nilpotent semigroup in which the product of any two subsemigroups coincides with their set-theoretic sum.

From Theorem 2 one can obtain a number of corollaries describing semigroups of one or another class with a distributive structure of subsemigroups. We note only one:

If Γ is a regular semigroup (see (2)) and $\Sigma'(\Gamma)$ is distributive, then Γ is a strong band of periodic locally cyclic groups.

Theorem 3. In order that $\Sigma(\Gamma)$ be a complemented structure, it is necessary and sufficient that Γ be the set-theoretic sum of its subsemigroups G and H, intersecting in the single idempotent e of the semigroup Γ, where G is a periodic group (with identity e) for which $\Sigma(G)$ is a complemented structure, and H is a semigroup with zero multiplication (with zero e, $H^2 = e$), and for any $g \in G$, $h \in H$ one has $gh = hg = g$.

The study of semigroups Γ for which $\Sigma'(\Gamma)$ is a complemented structure is apparently connected with great difficulties. However, under certain additional restrictions one can indicate the resulting classes of semigroups. We shall describe commutative semigroups of this kind.

Theorem 4. Let Γ be a commutative semigroup. In order that $\Sigma'(\Gamma)$ be a complemented structure, it is necessary and sufficient that Γ be a strong band of one-element semigroups.

Let us note that the sufficiency here remains valid even without the assumption that the semigroup is commutative. Moreover, if Γ is a strong band of one-element semigroups, then $\Sigma'(\Gamma)$ is even a structure with relative complements. The converse is not always true, although if $\Sigma'(\Gamma)$ is a structure with relative complements, then all elements of the semigroup Γ will be idempotents. We give an example.

Let Γ be the semigroup generated by the elements a and b with defining relations $a^2 = a$, $b^2 = b$, $aba = a$, $bab = b$. It is easy to see that Γ consists of the idempotents a, b, ab, and ba, and is not a strong band of its one-element subsemigroups, since $\{a, b\} = \Gamma \neq \{a\} \cup \{b\}$. But it is not hard to verify that $\Sigma'(\Gamma)$ is a structure with relative complements.

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Note: Figure translations are in progress. See original paper for figures.