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Abstract

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MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR A. V. POGORELOV

RIGIDITY OF CLOSED SURFACES NOT HOMEOMORPHIC TO THE SPHERE IN A RIEMANNIAN SPACE

Let a surface F in a Riemannian space R undergo a continuous deformation, passing at time t into the surface F_t. This deformation is called an infinitesimal bending if, at the initial instant ($t = 0$), the lengths of curves on the surface are stationary. With an infinitesimal bending of a surface there is naturally associated a vector field

$$\xi = \left.\frac{dx(t)}{dt}\right|_{t=0},$$

where $x(t)$ is the point of the surface F_t into which, under the deformation, the point x of the surface F passes. This vector field is called the bending field.

In the author' s work ([1]) it was proved that every closed surface homeomorphic to a sphere and with positive extrinsic curvature in a Riemannian space, fixed at one point together with the pencil of directions from this point, is rigid in the sense that every one of its bending fields is identically zero.

In the present note we consider the question of rigidity of closed surfaces not homeomorphic to the sphere and with positive extrinsic curvature. Such surfaces exist and are easily constructed in a nonsimply connected Riemannian space of negative curvature. For example, if the space contains a closed geodesic, then the locus of points of the space at a small distance ρ from this geodesic is a surface homeomorphic to a torus and with positive extrinsic curvature.

Theorem. *A closed surface homeomorphic to a torus and with positive extrinsic curvature in a Riemannian space, fixed at one point, is rigid; in particular, it admits no continuous bendings.*

Denote by F' the two-dimensional Riemannian manifold whose metric is given on the surface F by its second quadratic form. (Since the extrinsic curvature of

F is positive, the second quadratic form may be regarded as positive definite.) Map the manifold F' conformally onto some torus of zero curvature—a rectangle Δ_0 with opposite sides identified (the possibility of such a mapping is proved). Introduce in the plane of the rectangle Δ_0 a Cartesian coordinate system u, v with axes parallel to the sides of the rectangle. Assign, as curvilinear coordinates of a point on F, the Cartesian coordinates of the corresponding point of the rectangle Δ_0. Then the second quadratic form of the surface F will take the isothermal form $\lambda(du^2 + dv^2)$.

Denote by $\widetilde{F}$ the universal covering surface of F. The metric of the surface $\widetilde{F}$ is given by the same quadratic form as that of F inside the rectangle Δ_0 of the u, v-plane, and outside this rectangle is extended periodically. The bending field ξ, defined on F, we likewise extend to the surface $\widetilde{F}$. Introduce in a neighborhood of the surface $\widetilde{F}$ a semigeodesic coordinate system v^1, v^2, v^3, taking as the lines v^3 the geodesics perpendicular to $\widetilde{F}$, and as the surfaces $v^3 = \text{const}$ the surfaces equidistant from $\widetilde{F}$. As the parameters v^1, v^2, v^3 we take the signed distance of the point to the surface $\widetilde{F}$, v^3, and the coordinates u, v of the foot of the perpendicular.

of the geodesic perpendicular from the given point to the surface $\widetilde{F}(v^1, v^2)$. With our choice of parametrization of the surface and of the space, the equations for the covariant components ξ_1 and ξ_2 of the bending field ξ have the form

$$\frac{\partial \xi_1}{\partial u} - \frac{\partial \xi_2}{\partial v} = a\xi_1 + b\xi_2, \qquad \frac{\partial \xi_1}{\partial v} + \frac{\partial \xi_2}{\partial u} = c\xi_1 + d\xi_2,$$

where a, b, c, d are expressed in terms of the Christoffel symbols of the surface $\widetilde{F}$, and therefore are periodic functions of the variables u and v ([1]). Setting $\zeta = \xi_1 + i\xi_2$, $\overline{\zeta} = \xi_1 - i\xi_2$, $A = \frac{1}{4}(a + d + ic - ib)$, $B = \frac{1}{4}(a - d + ic + ib)$, $\partial/\partial\overline{z} = \frac{1}{2}(\partial/\partial u + i\partial/\partial v)$, one can give our system the following compact form:

$$\partial\zeta/\partial\overline{z} = A\overline{\zeta} + B\zeta.$$

The solution of this equation in the disk $|z| < R$ admits the representation

$$\zeta = \varphi(z)e^{\omega(z)},$$

where φ is some analytic function,

$$\omega(z) = \iint_{|z|<R} \frac{C(t)}{t - z}\, ds, \qquad C(t) = -\frac{1}{\pi}\left(A(t) + B(t)\frac{\overline{\zeta}(t)}{\zeta(t)}\right),$$

and the integration is carried out over the area of the disk ([2]).

Let q be a positive number less than one. Then, using the periodicity of the function $C(t)$, one can prove that in the disk $|z| \leq qR$

$$|\omega(z)| \leq c_0|z|,$$

where c_0 is a constant independent of R.

Now take the sequence of disks $K_n: \ |z| \leq n$ and construct a sequence of the corresponding functions $\omega_n(z)$ and $\varphi_n(z)$:

$$\zeta(z) = \varphi_n(z) e^{\omega_n(z)}.$$

Since the sequence of functions $\varphi_n(z)$ is bounded in every finite part of the u, v-plane, one can extract from it a convergent sequence. The limiting function $\varphi(z)$ will be analytic in the entire plane and grows no faster than $e^{c_0|z|}$.

By the condition of the theorem, at some point z_0 the function ζ vanishes (the point of fixation of the surface). In view of the periodicity, the function $\zeta(z)$ vanishes at the nodes of some lattice z_{mn}. At these same points the functions φ_n, and consequently also the limiting function φ, vanish. Since the function φ grows no faster than $e^{c_0|z|}$, being zero at points z_{mn}, it is identically zero. Consequently, the function $\zeta(z)$ is also identically zero.

Thus, for the bending field ξ the components ξ_1 and ξ_2 are equal to zero. As for the third component ξ_3, it vanishes together with ξ_1 and ξ_2, since

$$2\partial\xi_1/\partial u - \tilde{\Gamma}^k_{i0}\xi_k - 2\lambda\xi_3 = 0 \qquad (i, j, k = 1, 2)$$

(see ([1])). The theorem is proved.

In conclusion we note that the theorem is apparently true for a surface of any genus p, and, in the case $p > 1$, without any fixation conditions.

Physical-Technical Institute of Low Temperatures
Academy of Sciences of the Ukrainian SSR

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CITED LITERATURE

1. A. V. Pogorelov, *Some Questions of Geometry in the Large in Riemannian Space*, Kharkov, 1957.

2. I. N. Vekua, *Matem. sborn.*, **31**, No. 2 (1952).

Note: Figure translations are in progress. See original paper for figures.