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Abstract

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GENERAL FORM OF LINEAR DIFFERENTIAL EQUATIONS WHOSE ORDER IS REDUCED BY MEANS OF THE OPERATOR OF GENERALIZED DIFFERENTIATION D^α

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1. Introduction. The operator D^α, i.e. the integral transformation with the Euler-Cauchy kernel $(z-\zeta)^{-\alpha-1}$ ([1]), transforms a differential expression of order n of the form

$$\mathcal{L}[u] = \sum_{k=0}^{n} p_k(z) \frac{d^{n-k}u}{dz^{n-k}}, \tag{1}$$

where $p_k(z)$ are polynomials of degrees m_k, into the expression

$$T_\alpha[v] = \sum_{k=0}^{l} q_k(z,\alpha) \frac{d^{l-k}v}{dz^{l-k}}, \tag{2}$$

where

$$l = \max(m_0,\ m_1+1,\ m_2+2, \dots,\ m_n+n); \tag{3}$$

$$q_k(z,\alpha) = \sum_{h=0}^{\min(k,n)} \binom{\alpha}{k-h} p_h^{(k-h)}(z). \tag{4}$$

Between the expressions $\mathcal{L}$ and T_α there is the relation

$$D^\alpha \mathcal{L}[u] = T_\alpha[D^{\alpha+n-l}u] \quad ([2,3]). \tag{5}$$

Let

$$m_0 = n; \qquad m_k \le n-k \quad \text{for } k = 1, 2, \dots, n. \tag{6}$$

Then

$$l = n. \tag{7}$$

Putting

$$p_{n-k}(z) = \sum_{h=0}^{k} p_{n-k,h} z^{k-h}, \qquad q_{n-k}(z,\alpha) = \sum_{h=0}^{k} \Phi_{kh}(\alpha) z^h, \tag{8}$$

we find

$$\Phi_{k0}(\alpha) = \sum_{h_1=0}^{n-k} p_{n-k-h_1,k}\, \alpha(\alpha-1)\cdots(\alpha-h_1+1); \tag{9}$$

$$\Phi_{kh}(\alpha) = \frac{1}{h!}\Delta^h \Phi_{k-h,0}(\alpha), \tag{10}$$

where Δ^h is the finite difference of order h.

Under conditions (6), $z=\infty$ is a regular singular point of the equation

$$\mathcal{L}[u] = 0 \tag{11}$$

with characteristic exponents $\rho_1, \rho_2, \dots, \rho_n$, satisfying the determining equation

$$\Phi_{00}(\rho) = 0. \tag{12}$$

In the absence of logarithms, to the exponent ρ_k there corresponds a canonical solution $u_k(z)$ of equation (11) at $z=\infty$, expanded in a neighborhood of $z=\infty$ in the series

$$u_k(z) = \sum_{m=0}^{\infty} A_{km} z^{\rho_k - m}. \tag{13}$$

The operator $D^{\alpha}_{\infty z}$ is defined by the contour integral

$$D^{\alpha}_{\infty z} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i\left(1-e^{2(\alpha-\beta)\pi i}\right)} \int_C (\zeta - z)^{-\alpha-1} f(\zeta)\, d\zeta, \tag{14}$$

where $C = \zeta_0 z^+ \zeta_0 \infty^+ \zeta_0 z^- \zeta_0 \infty^- \zeta_0$, with ζ_0 an arbitrary regular point of the integrand, the loops $\zeta_0 \infty^{\pm} \zeta_0$ enclosing all singular points, and $\zeta_0 z^{\pm} \zeta_0$ only the singular point $\zeta = z$ of the integrand[4]. In this case

$$D_{\infty z}^{\alpha} z^{\beta} = e^{\pm \pi \alpha i} \frac{\Gamma(\alpha - \beta)}{\Gamma(-\beta)} z^{\beta - \alpha}. \tag{15}$$

To the canonical solutions (13) of equation (11) there correspond canonical solutions of the equation $T_{\alpha}[v] = 0$ of the form

$$v_k(z, \alpha) = D_{\infty z}^{\alpha} u_k(z) = e^{\pm \pi \alpha i} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha - \rho_k + m)}{\Gamma(m - \rho_k)} A_{km} z^{\rho_k - \alpha - m}. \tag{16}$$

If $\alpha = \rho_k - h$, where $h = 0, 1, 2, \ldots$, then $v_k \equiv \infty$, and

$$^{*}v_k(z) = \lim_{\alpha \to \rho_k - h} \frac{v_k(z, \alpha)}{\Gamma(\alpha - \rho_k)} =$$

$$= e^{\pm \pi (\rho_k - h) i} \sum_{m=0}^{h} (-1)^m \frac{h(h-1) \ldots (h - m + 1)}{\Gamma(m - \rho_k)} A_{km} z^{h-m}. \tag{17}$$

2. In the present paper we investigate the properties of those expressions $\mathcal{L}[u]$ whose order is lowered by means of the operator D^{α}. If, under conditions (6), the relations

$$q_n(z, \alpha) \equiv 0, \qquad q_{n-1}(z, \alpha) \equiv 0, \ldots, \qquad q_{n-\mu+1}(z, \alpha) \equiv 0 \tag{18}$$

hold for any z and fixed $\alpha = \alpha_1$, then, putting

$$\Lambda[w] = \sum_{k=0}^{n-\mu} q_k(z, \alpha_1) \frac{d^{\,n-\mu-k} w}{dz^{\,n-\mu-k}}, \tag{19}$$

we obtain

$$D^{\alpha_1} \mathcal{L}[u] = \Lambda[D^{\alpha_1 + \mu} u]. \tag{20}$$

In this case the operator D^{α_1} transforms the expression of n-th order $\mathcal{L}[u]$ into the expression of $(n - \mu)$-th order $\Lambda[w]$, where $w = D^{\alpha_1 + \mu} u$.

Theorem. Let there be given natural numbers n and $\mu < n$, polynomials $p_0(z), p_1(z), \ldots, p_{n-\mu}(z)$ of degrees $n, n-1, \ldots, \mu$, and a complex number ω_1

Then equations (18) for $a = a_1$ uniquely determine the polynomials $p_{n-\mu+1}(z), \ldots, p_n(z)$. The corresponding expression $\mathcal{L}[u]$ satisfies relation (20), i.e., the operator D^{α_1} lowers the order of $\mathcal{L}[u]$ by μ units. Equations (18) for $a = a_1$ are equivalent to the equations

$$\Phi_{00}(\alpha_1) = 0; \qquad \Phi_{00}(\alpha_1 + 1) = 0;\ \dots;\ \Phi_{00}(\alpha_1 + \mu - 2) = 0; \tag{21}$$

$$\Phi_{00}(\alpha_1 + \mu - 1) = 0;$$

$$\Phi_{10}(\alpha_1) = 0; \qquad \Phi_{10}(\alpha_1 + 1) = 0;\ \dots;\ \Phi_{10}(\alpha_1 + \mu - 2) = 0;$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\Phi_{\mu-2,0}(\alpha_1) = 0; \qquad \Phi_{\mu-2,0}(\alpha_1 + 1) = 0; \tag{22}$$

$$\Phi_{\mu-1,0}(\alpha_1) = 0.$$

Equations (21) mean that the order α_1 of the operator D^{α_1}, as well as the numbers $\alpha_1 + 1, \dots, \alpha_1 + \mu - 1$, are characteristic exponents of equation (11) at $z = \infty$. Suppose that, for any natural number h,

$$\Phi_{00}(\alpha_1 - h) \neq 0. \tag{23}$$

Then from conditions (22) it follows that to the exponents $\rho_k = \alpha_1 + k - 1$ $(k = 1, 2, \dots, \mu)$ there belong μ linearly independent logarithm-free solutions $u_k(z)$ of equation (11) of the form (13).

3. Since for $1 \leq k \leq \mu$, $\rho_k - \alpha_1 = k - 1$, the solutions $v_1(z, \alpha_1), \dots, v_\mu(z, \alpha_1)$ of the equation $T_{\alpha_1}[v] = 0$ lose their meaning and must be replaced by the solutions

$$v_k^*(z) = \lim_{\alpha \to \alpha_1} \frac{v_k(z, \alpha)}{\Gamma(\alpha - \alpha_1 - k + 1)},$$

where $v_k^*(z)$ is a polynomial of degree $k-1$. If the differences $\rho_{\mu+1} - \alpha_1, \dots, \rho_n - \alpha_1$ are not integers, then the functions

$$w_{\mu+1} = D_{\infty z}^{\alpha_1 + \mu} u_{\mu+1}(z), \dots, w_n(z) = D_{\infty z}^{\alpha_1 + \mu} u_n(z) \tag{24}$$

form a fundamental system of the equation

$$\Lambda[w] = 0. \tag{25}$$

On the other hand, for $1 \leq k \leq \mu$,

$$\lim_{\alpha\to\alpha_1}\sum_{j=0}^{\mu-1} q_{n-j}(z,\alpha)\,\frac{d^j v_k(z,\alpha)}{dz^j} = Q_k(z),$$

where $Q_k(z)$ is a polynomial of degree $k-1$, so that the functions $w_k = D_{\infty z}^{\alpha_1+\mu} u_k$ for $k \leq \mu$ satisfy the equations $\Lambda[w_k] = -Q_k$, whence it follows that all n functions $w_i = D_{\infty z}^{\alpha_1+\mu} u_i(z)$ $(i = 1, 2, \ldots, n)$ form a fundamental system of the equation

$$\frac{d^\mu}{dz^\mu}\Lambda[w] = 0. \tag{26}$$

The fundamental system of the equation

$$D^\alpha \mathcal{L}[u] = 0 \tag{27}$$

consists of the functions

$$1,\ z,\ z^2, \ldots, z^{\mu-1},\ D_{\infty z}^{\alpha_1} u_{\mu+1}(z), \ldots, D_{\infty z}^{\alpha_1} u_n(z).$$

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References Cited

1. E. L. Ince, *Ordinary Differential Equations*, Ch. VIII, 1939.

2. N. Ya. Sonin, *Matem. sborn.*, 6, 1 (1872).

3. P. A. Nekrasov, *Matem. sborn.*, 14 (1888).

4. R. Courant, D. Hilbert, *Methods of Mathematical Physics*, 2, 1951, pp. 520 –529.

Note: Figure translations are in progress. See original paper for figures.