MATHEMATICS

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. G. LEMLEIN

SPACES WITH CENTRO-PROJECTIVE CONNECTION $(\Gamma^i_{jk}, \Gamma_{lm})$ AS MANIFOLDS IMMERSED IN THE REPRESENTATION SPACE OF THE PROLONGED PSEUDOGROUP OF ANALYTIC TRANSFORMATIONS

(Presented by Academician P. S. Aleksandrov, 15 II 1961)

In this article an invariant construction is given of a space with centro-projective connection $(\Gamma^i_{jk}, \Gamma_{lm})$ from a differential neighborhood of the third order, starting from the system of exterior differential equations of É. Cartan for the prolonged pseudogroup of analytic transformations ([2]):

$$D\omega^i = [\omega^l \omega^i_l], \qquad D\omega^i_j = [\omega^n_j \omega^i_m] + [\omega^i_{jm}\omega^m],$$
$$D\omega^j_{ik} = [\omega^m_{jk}\omega^i_m] - [\omega^i_{km}\omega^m_j] - [\omega^i_{jm}\omega^m_k] + [\omega^i_{jkm}\omega^m]. \tag{1}$$

All forms occurring here are symmetric in any pair of their lower indices. The foundations of the algebraic method used, based on the theory of successive prolongations and of enclosures of some geometric objects by others, were developed by G. F. Laptev and A. M. Vasiliev and set forth in the papers ([3], [1]).

The local coordinates of the initial manifold $\{V^n\}$ are given by the first integrals of the completely integrable Pfaff system $\omega^i = 0$, and the pseudogroup of their analytic transformations $x^{i'} = x^{i'}(x^i)$ is defined by the equations $\Omega^i = \omega^i$. Here and in what follows, capital letters denote forms obtained from the forms denoted by the corresponding lowercase letters by simply replacing the transformed variables by the transformed ones.

Contracting in relations (1) the forms ω^i_j and ω^i_{jk} over the upper and lower indices, we obtain

$$D\omega^l_l = [\omega^l_{lm}\omega^m], \qquad D\omega^l_{lk} = [\omega^m_k \omega^l_{lm}] + [\omega^l_{lkm}\omega^m]. \tag{2}$$

Introducing now, alongside the forms of affine connection,

$$\tilde{\omega}^k_i = \omega^k_i + \Gamma^k_{im}\omega^m, \tag{3}$$

the forms

$$\tilde{\omega}_{lk}^{l} = \omega_{lk}^{l} + (n+1)\Gamma_{km}\omega^{m}, \tag{4}$$

we shall have

$$D\tilde{\omega}_{i}^{k} - [\tilde{\omega}_{i}^{m}\tilde{\omega}_{m}^{k}] = [\vartheta_{im}^{k}\omega^{m}], \tag{5}$$

where

$$\vartheta_{im}^{k} \equiv d\Gamma_{im}^{k} - \Gamma_{ij}^{k}\omega_{m}^{j} - \Gamma_{im}^{k}\omega_{i}^{j} + \Gamma_{im}^{j}\omega_{j}^{k} + \omega_{im}^{k} - \Gamma_{il}^{j}\Gamma_{jm}^{k}\omega^{l},$$

$$D\tilde{\omega}_{lk}^{l} - [\tilde{\omega}_{k}^{m}\tilde{\omega}_{lm}^{l}] = (n+1)[\vartheta_{km}\omega^{m}], \tag{6}$$

$$\vartheta_{km} \equiv d\Gamma_{km} - \Gamma_{kj}\omega_{m}^{j} - \Gamma_{jm}\omega_{k}^{j} + \frac{1}{(n+1)}\Gamma_{km}^{j}\omega_{aj}^{a} + \frac{1}{(n+1)}\omega_{akm}^{a} - \Gamma_{jm}\Gamma_{kl}^{j}\omega^{l}.$$

Let us consider the $(n+n^2+n^3)$-dimensional representation space of such a prolongation of the pseudogroup of analytic transformations which shows how the object of centro-projective connection transforms. The points of such a space $(x^i, \Gamma_{jk}^{i}, \Gamma_{lm})$ are given by the first integrals of a completely ···

of the integrable Pfaff system $\omega^i = 0$, $\vartheta_{jk}^{i} = 0$, $\vartheta_{lm} = 0$, and the pseudogroup transforming them:

$$x^{i'} = x^{i'}(x^i), \qquad \Gamma_{j'k'}^{i'} = \frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial x^{j}}{\partial x^{j'}}\frac{\partial x^{k}}{\partial x^{k'}}\Gamma_{jk}^{i} + \frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial^2 x^{i}}{\partial x^{j'}\partial x^{k'}},$$

$$\Gamma_{l'm'} = \frac{\partial x^{l}}{\partial x^{l'}}\frac{\partial x^{m}}{\partial x^{m'}}\left[\Gamma_{lm} - \left(\frac{\partial^2 \ln\det\left\|\partial x^{r'}/\partial x^{r}\right\|^{-1/(n+1)}}{\partial x^{l}\partial x^{m}} - \frac{\partial \ln\det\left\|\partial x^{r'}/\partial x^{r}\right\|^{-1/(n+1)}}{\partial x^{k}}\Gamma_{lm}^{k}\right)\right],$$

is determined by the equations $\Omega^i = \omega^i$, $\Theta_{jk}^{i} = \vartheta_{jk}^{i}$, $\Theta_{lm} = \vartheta_{lm}$.

A space with centro-projective connection $(\Gamma_{jk}^{i}, \Gamma_{lm})$ from a differential neighborhood of third order is now defined as an n-dimensional surface

$$\Gamma_{jk}^{i} = \Gamma_{jk}^{i}(x^m), \qquad \Gamma_{kl} = \Gamma_{kl}(x^m), \tag{7}$$

immersed in the $(n+n^2+n^3)$-dimensional space under consideration. Instead of specifying an n-dimensional surface by equations (7), one may evidently simply specify the forms ϑ^i_{jk} and ϑ_{kl} in the form of linear combinations of the forms ω^m:

$$\vartheta^i_{jk} = \Lambda^i_{j,km}\omega^m, \qquad \vartheta_{kl} = \Lambda_{k,lm}\omega^m. \tag{8}$$

Substituting (3) and (4) into the first equations of the systems (1) and (2), and (8) into (5) and (6), we obtain the structure equations of the space under study in the form

$$D\omega^i = [\omega^k\tilde{\omega}^i_k] - R^i_{km}[\omega^k\omega^m], \qquad D\omega^l_i = [\tilde{\omega}^l_{im}\omega^m] + (n+1)R_{km}[\omega^k\omega^m], \tag{9}$$

where

$$R^i_{km} = \frac{1}{2}(\Gamma^i_{km} - \Gamma^i_{mk}), \qquad R_{km} = \frac{1}{2}(\Gamma_{km} - \Gamma_{mk}); \tag{10}$$

$$D\tilde{\omega}^i_j = [\tilde{\omega}^m_j\tilde{\omega}^i_m] - \frac{1}{2}R^i_{j,km}[\omega^k\omega^m], \tag{11}$$

$$D\tilde{\omega}^l_{il} = [\tilde{\omega}^m_i\tilde{\omega}^l_{lm}] - \frac{1}{2}(n+1)R_{i,km}[\omega^k\omega^m],$$

$$R^i_{j,km} = \Lambda^i_{j,km} - \Lambda^i_{j,mk}, \qquad R_{i,km} = \Lambda_{i,km} - \Lambda_{i,mk}. \tag{12}$$

The quantities (10) and (12) satisfy, for $\omega^i = 0$, the equations

$$dR^i_{km} - R^i_{lm}\tilde{\omega}^l_k - R^i_{kl}\tilde{\omega}^l_m + R^l_{km}\tilde{\omega}^i_l = 0, \tag{13}$$

$$dR_{km} - R_{lm}\tilde{\omega}^l_k - R_{kl}\tilde{\omega}^l_m + \frac{1}{(n+1)}R^l_{km}\tilde{\omega}^i_{il} = 0;$$

$$dR^i_{j,km} - R^i_{j,kl}\tilde{\omega}^l_m - R^i_{j,lm}\tilde{\omega}^l_k - R^i_{l,km}\tilde{\omega}^l_j + R^l_{j,km}\tilde{\omega}^i_l = 0, \tag{14}$$

$$dR_{i,km} - R_{i,kl}\tilde{\omega}^l_m - R_{i,lm}\tilde{\omega}^l_k - R_{l,km}\tilde{\omega}^l_i + \frac{1}{(n+1)}R^l_{i,km}\tilde{\omega}^j_{jl} = 0,$$

which form completely integrable systems and, consequently, are geometric objects.

The first equations of the systems (10) and (12), by virtue of (13) and (14), define the torsion and curvature tensors of the space of affine connection specified by the first equations of the system (8). The structure equations of this space are determined by the first equations of the systems (9) and (11). We note that when $R^i_{km} = 0$ and $R^i_{j,km} = 0$, the second equations of (10) and (12) give, as is directly seen from the systems (13) and (14), tensors. The geometric objects (10) and (12) are called, respectively, the complete torsion object and the complete curvature object. Under a change of coordinates these objects are transformed according to the laws

$$R^{i'}_{j'k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}R^i_{jk}, \qquad R_{k'm'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\left(R_{km} + \frac{\partial \ln\det\left\|\partial x^{r'}/\partial x^r\right\|^{-1/(n+1)}}{\partial x^l}R^l_{km}\right),$$

$$R^{i'}_{j',\,k'm'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}R^i_{j,\,km},$$

$$R_{i',\,k'm'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\left(R_{i,\,km} + \frac{\partial \ln\det\|\partial x^{r'}/\partial x^r\|^{-1/(n+1)}}{\partial x^l}R^l_{i,\,km}\right).$$

The points of the local centro-projective spaces $\{P^n\}$ (4) are specified by geometric objects (u^i) (punctors), determined by the completely integrable system

$$\vartheta^i \equiv du^i + u^l\tilde{\omega}^i_l - \frac{1}{(n+1)}u^i u^l\tilde{\omega}^m_{ml} = 0, \qquad \omega^i = 0.$$

Integrating the system $\Omega^i = \omega^i$, $\Theta^i = \vartheta$, we obtain the transformation law for (u_i) in finite form

$$u^{i'} = \frac{(\partial x^{i'}/\partial x^i)u^i}{-\frac{1}{(n+1)}\frac{\partial \ln\det\|\partial x^{r'}/\partial x^r\|}{\partial x^k}u^k + 1}.$$

The points of the space $\{Q^n\}$, dual to $\{P^n\}$ (6), are specified by geometric objects (u_i) (copunctors), determined by the completely integrable system

$$\vartheta_i \equiv du_i - u_l\tilde{\omega}^l_i - \frac{1}{(n+1)}\tilde{\omega}^m_{mi} = 0, \qquad \omega^i = 0.$$

Integrating the system $\Omega^i = \omega^i$, $\Theta_i = \vartheta_i$, we obtain the transformation law for (u_i) in finite form

$$u_{i'} = \frac{\partial x^i}{\partial x^{i'}}u_i - \frac{1}{(n+1)}\frac{\partial \ln\det\|\partial x^r/\partial x^{r'}\|}{\partial x^{i'}}.$$

We now introduce the connection $\tilde{\Gamma}^i_{jk}$ into the local centro-projective spaces $\{P^n\}$. Externally differentiating the forms

$$\vartheta^i \equiv du^i + u^l \tilde{\omega}^i_l - \frac{1}{(n+1)} u^i u^l \tilde{\omega}^m_{ml},$$

whose first integrals determine a point in $\{P^n\}$, and using formulas (11) with $\omega^i = 0$, we obtain

$$D\vartheta^i = [\vartheta^l \vartheta^i_l], \tag{15}$$

where

$$\vartheta^i_l \equiv \tilde{\omega}^i_l - \frac{1}{(n+1)} u^k (\delta^i_k \tilde{\omega}^m_{ml} + \delta^i_l \tilde{\omega}^m_{mk}). \tag{16}$$

Next, externally differentiating the forms (16) and again using (11) with $\omega^i = 0$, we have

$$D\vartheta^i_l = [\vartheta^m_l \vartheta^i_m] + [\vartheta^i_{lm} \vartheta^m], \tag{17}$$

where

$$\vartheta^i_{lm} \equiv \frac{1}{(n+1)} (\delta^i_l \tilde{\omega}^a_{am} + \delta^i_m \tilde{\omega}^a_{al}). \tag{18}$$

From (16), (18) it is easy to obtain $\vartheta^a_a = \tilde{\omega}^a_a - u^k \tilde{\omega}^a_{ak}$, $\vartheta^a_{am} \equiv \tilde{\omega}^a_{am}$; consequently,

$$\vartheta^i_{lm} \equiv \frac{1}{(n+1)} (\delta^i_l \vartheta^a_{am} + \delta^i_m \vartheta^a_{al}). \tag{19}$$

Introducing now in $\{P^n\}$ the forms of an affine connection

$$\tilde{\vartheta}^k_i = \vartheta^k_i + \tilde{\Gamma}^k_{im} \vartheta^m,$$

we shall have

$$D\tilde{\vartheta}^k_i - [\tilde{\vartheta}^m_i \tilde{\vartheta}^k_m] = [\varphi^k_{im} \vartheta^m], \tag{20}$$

where

$$\varphi^k_{im} \equiv d\tilde{\Gamma}^k_{im} - \tilde{\Gamma}^k_{ij} \vartheta^j_m - \tilde{\Gamma}^k_{jm} \vartheta^j_i + \tilde{\Gamma}^j_{im} \vartheta^k_j + \vartheta^k_{im} - \tilde{\Gamma}^j_{il} \tilde{\Gamma}^k_{jm} \vartheta^l.$$

The equations defining in the space $\{P^n\}$ the object of affine connection $\tilde{\Gamma}^k_{im}$ are written in the form

$$d\tilde{\Gamma}^k_{im} - \tilde{\Gamma}^k_{ij}\tilde{\vartheta}^j_m - \tilde{\Gamma}^k_{jm}\tilde{\vartheta}^j_i + \tilde{\Gamma}^j_{im}\tilde{\vartheta}^k_j + \vartheta^k_{im} = 0$$

and, consequently, by virtue of (19), coincide with the equations

$$d\tilde{\gamma}^k_{im} - \tilde{\gamma}^k_{ij}\tilde{\vartheta}^j_m - \tilde{\gamma}^k_{jm}\tilde{\vartheta}^j_i + \tilde{\gamma}^j_{im}\tilde{\vartheta}^k_j + \frac{1}{(n+1)}\left(\delta^k_i\vartheta^l_{lm} + \delta^k_m\vartheta^l_{li}\right) = 0,$$

which are satisfied by the object ([5],[7])

$$\tilde{\gamma}^k_{im} = -\delta^k_i\tilde{a}_m - \delta^k_m\tilde{a}_i, \tag{21}$$

where

$$\tilde{a}_i = \frac{\left[\frac{1}{(n+1)^2}\Gamma^a_{ai}\Gamma^b_{bk} - \frac{\sigma_{ik}}{(n-1)}\right]u^k - \frac{1}{(n+1)}\Gamma^a_{ai}}{\left[\frac{1}{(n+1)^2}\Gamma^a_{aj}\Gamma^b_{bk} - \frac{\sigma_{jk}}{(n-1)}\right]u^ju^k - \frac{2}{(n+1)}\Gamma^a_{aj}u^j + 1}, \tag{22}$$

$$\sigma_{jk} = \frac{1}{2}\left[R^l_{j,kl} + R^l_{k,jl}\right].$$

Thus, the object (21) may be taken in $\{P^n\}$ as an object of affine connection. This connection transforms the space $\{P^n\}$ into a symmetric projective-Euclidean space ([8]).

Substituting $\vartheta^i_l = \tilde{\vartheta}^i_l - \tilde{\gamma}^i_{lm}\vartheta^m$ into (15) and $\varphi^i_{jk} = (\partial\tilde{\gamma}^i_{jk}/\partial u^m - \tilde{\gamma}^a_{jm}\tilde{\gamma}^i_{ak})\vartheta^m$ into (20), we obtain the structure equations of this space in the form

$$D\vartheta^i = [\vartheta^l\tilde{\vartheta}^i_l], \qquad D\tilde{\vartheta}^i_j = [\tilde{\vartheta}^l_j\tilde{\vartheta}^i_l] - \frac{1}{2}\widetilde{R}^i_{j,km}[\vartheta^k\vartheta^m],$$

where the components

$$\widetilde{R}^i_{j,km} = \frac{\partial\tilde{\gamma}^i_{mj}}{\partial u^k} - \frac{\partial\tilde{\gamma}^i_{kj}}{\partial u^m} + \tilde{\gamma}^a_{mj}\tilde{\gamma}^i_{ak} - \tilde{\gamma}^a_{kj}\tilde{\gamma}^i_{am} \tag{23}$$

form the curvature tensor. Substituting (22) into (21), and (21) into (23), we obtain

$$\widetilde{R}^i_{j,km} = \frac{1}{(1-n)}\left(\delta^i_k\widetilde{R}_{mj} - \delta^i_m\widetilde{R}_{kj}\right),$$

where

$$\widetilde{R}_{k,l} = (1-n)\frac{\frac{1}{(n+1)^2}\Gamma^a_{ak}\Gamma^b_{bl} - \frac{\sigma_{kl}}{(n-1)}}{\left[\frac{1}{(n+1)^2}\Gamma^a_{ai}\Gamma^b_{bj} - \frac{\sigma_{ij}}{(n-1)}\right] u^{iu^j} - \frac{2}{(n+1)}\Gamma^a_{ai}u^i + 1} - \frac{(1-n)\left[\left(\frac{1}{(n+1)^2}\Gamma^a_{ai}\Gamma^b_{bk} - \frac{\sigma_{ik}}{(n-1)}\right) u^i - \cdots\right.}{\left[\left(\frac{1}{(n+1)^2}\Gamma^a_{ai}\Gamma^b_{bj} - \cdots\right.\right.}$$

Since, by virtue of (24), $R_{k,l}\big|_{u=0} = \sigma_{kl}$, and consequently

$$\widetilde{R}^i_{j,km}\big|_{u=0} = \frac{1}{(1-n)}\left(\delta^i_k\sigma_{mj} - \delta^i_m\sigma_{kj}\right),$$

then for the tensor of equiprojective curvature ([7])

$$T^h_{k,ji} = R^h_{k,ji} + \frac{1}{(n-1)}\left(\delta^h_j\sigma_{ik} - \delta^h_i\sigma_{jk}\right),$$

whose vanishing is a necessary and sufficient condition that the space under consideration be an equiaffine projective-Euclidean space, we obtain

$$T^h_{k,ji} = R^h_{k,ji} - \widetilde{R}^h_{k,ji}\big|_{u=0}.$$

Moscow State Pedagogical Institute
named after V. I. Lenin

Received
14 II 1961

CITED LITERATURE

[1] A. M. Vasil' ev, DAN, 79, No. 1 (1951). [2] E. Cartan, Ann. Ec. Norm. Sup., 21 (1904); Sup. 22 (1905); Sup. 25 (1908). [3] T. F. Laptev, Tr. Mosk. matem. obshch., 2, 275 (1953). [4] V. G. Lemlein, DAN, 129, No. 2 (1959). [5] V. G. Lemlein, DAN, 131, No. 1 (1960). [6] V. G. Lemlein, DAN, 132, No. 6 (1960). [7] V. G. Lemlein, DAN, 133, No. 6 (1960). [8] P. A. Shirokov, Tr. seminara po vektorn. i tenzorn. analizu, vol. 8, 73 (1950).

Note: Figure translations are in progress. See original paper for figures.